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Supply function equilibria in transportation networks*

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Transport constraints limit competition and arbitrageurs' possibilities of exploiting price differences between commodities in neighbouring markets.We analyze a transportation network where oligopoly producers compete with supply functions under uncertain demand, as in wholesale electricity markets. For symmetric networks with a radial structure, we show that existence of symmetric supply function equilibria (SFE) is ensured if demand shocks are sufficiently evenly distributed. We can explicitly solve for them for uniform multi-dimensional nodal demand shocks.

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1 Introduction

The transport of commodities can be conducted via air, road or rail routes (in case of freight) or through pipelines (in case of gas or oil), or transmission lines (in case of electricity). The passage of the commodity through these conduits is typically represented in economic models using a flow network [37]. These have *nodes* representing junctions and *arcs* representing transport routes. The flow through each arc is limited by its transport capacity.

Transport constraints limit trade, which makes production and consumption less efficient. Moreover, transport constraints reduce competition between agents situated in separated markets, which worsens market efficiency even further. Congestion is of particular importance for markets with negligible storage possibilities, such as wholesale electricity markets. Then demand and supply must be instantly balanced and temporary congestion in the network can result in large local price spikes. The same market can at times exhibit very little market power and, at other times, suffer from the exercise of a great deal of market power. Borenstein et al. [13] show that standard concentration measures such as the Herfindahl-Hirschman index (HHI) work poorly to assess the degree of competition in such markets. Thus competition authorities who need to predict the use of market power under various counterfactuals – what might happen if a merger or acquisition is accepted or transport capacity is expanded, need more detailed analytical tools.

We analyze the influence a network's topology and transport constraints have on competition in an oligopoly market with a homogeneous commodity. Producers and consumers are located in nodes of the network, which are connected by arcs (lines). Transports in an arc are costless up to its transport capacity. We focus on *connected radial* networks, where there is a unique path (chain of arcs) between every two nodes in the network. We say that two nodes are completely integrated when they are connected via uncongested arcs. A node is always completely integrated with itself.

We assume that the commodity is traded in a network where each node has a local market price. Consumers are price-takers and production costs are common knowledge. We consider a simultaneous-move game, where each strategic producer first commits to a supply function, as in a multi-unit auction, and then a local exogenous additive demand shock is realized in each node of the network. After the demand shocks have been realized, the commodity is transported along arcs between the nodes by price-taking arbitrageurs or a regulated, price-taking network operator, who buy the commodity in one node and sell it in another node. The market is cleared when all feasible arbitrage opportunities have been exhausted by the transport sector. We solve for a Nash equilibrium of supply functions, also called a *Supply Function Equilibrium* (SFE).

The slope of the residual demand curve[1] is important in the calculation of a producer's optimal offer. Even if competitors play pure-strategies, the slope of the

[1] The residual demand at a specific price is given by demand at that price less competitors' sales as that price.

residual demand curve is uncertain in equilibrium due to the local demand shocks. We characterize this uncertainty using Anderson and Philpott's [4] *market distribution function* approach, which is analogous to Wilson's probability distribution of the market price [46]. For radial networks, we show that the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand curve that it is facing.[2] In the paper we use graph theory to show how the expected slope of the residual demand curve relates to the characteristics of the network and competitors' supply functions. For a given vector of demand shocks, the flow through arcs with a strictly binding transport capacity is fixed on the margin. Thus on the margin, a firm's realized residual demand is only influenced by production and consumption in nodes that are completely integrated with the firm's node. Due to the arbitrageurs in the transport sector, two nodes that are completely integrated will have the same market price. We define a *market integration function* for each producer, which equals the expected number of nodes that the producer's node is completely integrated with (including its own node) conditional on the producer's output and its local market price.

In principle, a system of our optimality conditions can be used to numerically solve for Supply Function Equilibria in general networks. In our paper we use them to solve for symmetric equilibria in two-node and star networks with inelastic demand.[3] Firms' supply functions depend on the number of firms in the market and generally network flows depend on firms' supply functions. Still it can be shown that in a symmetric equilibrium with inelastic demand, market integration can be determined from the following parameters: the network topology, the demand shock distribution, transport capacities and nodal production capacities. In this case equilibrium flows do not depend on the number of firms. The implication is that oligopoly producers will have high mark-ups at output levels for which the market integration function, which can be calculated for a competitive market, returns small values, and lower mark-ups at output levels where market integration is expected to be high.

In the special case of multi-dimensional uniformly distributed demand shocks, the market integration function simplifies to a constant for symmetric equilibria. We show that a producer's equilibrium supply function in a node of such a network is identical to an equilibrium supply function in an isolated node where the number of symmetric firms have been scaled by the market integration function. Thus previous analytical results for symmetric SFE in single node networks and properties of such SFE [5][21][25][29][38] can be generalized to symmetric SFE in symmetric radial networks with transport constraints and multi-dimensional uniformly distributed demand shocks.

We focus on characterising supply function equilibrium (SFE) in radial networks. However, we also show how our optimality conditions can be generalized to consider *meshed* networks, which have multiple paths (*loop flows*) between some

[2] Note that the output of the firm influences congestion in the network, which in its turn influences its residual demand curve. Thus with the slope of a firm's residual demand we here mean the slope of its residual demand conditional on that it has a fixed output.

[3] There are no strategic producers in the center node of the star network. Thus the network is symmetric from the producers' perspective.

nodes in the network. Although the meshed model does not simplify as in the radial case. Moreover, we describe how our conditions can be modified to calculate SFE in networks with a discriminatory multi-unit auction (pay-as-bid pricing) in the nodes and Cournot Nash equilibrium in networks with uncertain demand. Normally nodes represent the geographical location of a market place, and with transport we normally mean that the commodity is moved from one geographical location to another location. But nodes and transports could be interpreted in a more general sense. For example, a node could represent a point in time or a geographical location at a particular point in time. Moreover, storage at a geographical location can be represented by arcs that allow for transports of the commodity to the same location, but at a later point in time. The transport capacity of such arcs would then correspond to the local storage capacity.

The supply function equilibrium for a single node with marginal pricing was originally developed by Klemperer and Meyer [29]. This model represents a generalized form of competition in oligopoly markets, in-between the extremes of the Bertrand and Cournot equilibrium. The setting of the SFE is particularly well-suited for markets where producers submit supply functions to a uniform-price auction before demand has been realized, as in wholesale electricity markets [12][21][25]. This has also been confirmed qualitatively and quantitatively in several empirical studies of bidding in electricity markets.[4] But the SFE model is of more general interest. Klemperer and Meyer [29] argue that although most markets are not explicitly cleared by uniform-price auctions, firms typically face a uniform market price and they need predetermined decision rules for lower-level managers to deal with changing market conditions. Thus, in general, firms implicitly commit to supply functions. Klemperer and Meyer's model has only one uncertain parameter, a demand shock. In equilibrium there is a one to one mapping between the price and shock. Thus each firm can choose its supply function such that its output is optimal for every realized shock. Klemperer and Meyer's equilibrium is therefore said to be *ex-post* optimal. As noted by Anderson et al [6], this feature is difficult to translate into a network with multi-dimensional demand shocks.[5]

By requiring that each firm's offer is optimal only in expectation, the recent paper by Wilson [45] takes a different approach, which enables him to extend Klemperer and Meyer's [29] model to consider the network's influence on bidding strategies.[6] Wilson, however, does not derive any second-order conditions in his

[4]Empirical studies of the electricity market in Texas (ERCOT) show that supply functions of the two to three largest firms in this market roughly match Klemperer and Meyer's first-order condition [27][40]. The fit is worse for small producers. According to Wolak [47] the reason is that these studies did not consider that supply functions are stepped. He shows that both large and small electricity producers in Australia choose stepped supply functions in order to maximize profits; at least observed data does not reject this hypothesis.

[5]Anderson et al [6] investigate a two-node transmission network with both independent and correlated demand at the nodes. They derive formulae to represent the market distribution function for a producer when its network becomes interconnected to a previously separate grid under the assumption that the interconnection does not change competitors' supply functions.

[6]Lin and Baldick [31] and Lin et al [32] also calculate first-order conditions for transmission networks with supply function offers, but their model is limited to cases with certain demand.

paper nor does he solve for SFE, so his analysis is missing some fundamental components that our analysis provides.

Previous research has shown that second-order conditions are often violated in a network with strategic producers. The reason is that transport constraints can introduce kinks (nonsmoothness) in a producer's residual demand curve which becomes discontinuously less price sensitive when net imports to its node are congested. Thus, in a node where imports are nearly congested it will be profitable for a producer to withhold production in order to push the price above the next breakpoint in its residual demand curve. This type of deviation will often rule out pure-strategy Nash equilibria.[7] Borenstein et al. [14] for example rule out Cournot NE when the transport capacity between two symmetric markets is sufficiently small and demand is certain. Downward et al. [17] analyse existence of Cournot equilibria in general networks with transport constraints.[8] We verify that monotonic solutions to our first-order conditions are Supply Function Equilibria (SFE) when the shock density is sufficiently evenly distributed, i.e. close to a uniform multi-dimensional distribution. In this case the demand shocks will smooth the residual demand curve, so that its breakpoints disappear in expectation. But existence of SFE cannot be taken for granted. Perfectly correlated demand shocks or steep slopes and discontinuities in the demand shock density will not smooth the residual demand curve sufficiently well, and then profitable deviations from the first-order solution will exist.[9]

Our paper also differs from Wilson [45] in the source of randomness. In his proofs, local demand is certain in all markets but one, and transmission capacities are uncertain. Nevertheless, even if our calculations are less straightforward, especially for meshed networks, we find it important to also analyze the multi-market case with local net-demand shocks/variations and known transmission capacities. We believe that our model is of particular relevance for markets with long-lived bids, such as PJM[10], where producers' offers are fixed during the whole day to meet a wide range of local demand outcomes. Also large local net-demand shocks can occur on short notice, especially in electricity networks with significant amounts of wind power, so our model is also relevant for markets with short lived bids.

Our model is mainly intended to represent competition in a transportation net-

[7] But Escobar and Jofre [18] show that there is normally a mixed-strategy NE in those cases. Adler et al. [1] and Hu and Ralph [28] show that existence of pure-strategy Cournot NE depends on the assumptions made about the rationality of the players. Hobbs et al [22] bypasses the existence issue by using conjectural variations instead of a Nash equilibrium. Existence of equilibria is more straightforward in networks with infinitesimally small producers [16][18][26].

[8] Willems [44] analyse how a network operator's rule to allocate transmission capacity influences the Cournot NE. Wei and Smeers [43] calculate Cournot NE in transmission networks with regulated transmission prices. Oren [36] calculates Cournot NE in a network with transmission rights. Neuhoff et al's [34] use Cournot NE to analyse competition in the northwestern European wholesale electricity market.

[9] Note that a discontinuity in a node's shock density is acceptable as long as it occurs when transport capacities in all arcs to the node are binding.

[10] PJM is the largest existing deregulated wholesale electricity market. Originally PJM coordinated the movement of wholesale electricity in Pennsylvania, New Jersey and Maryland. Now PJM has expanded to also cover all or parts of Delaware, Illinois, Indiana, Kentucky, Michigan, North Carolina, Ohio, Tennessee, Virginia, West Virginia and the District of Columbia.

work with tangible objects, such as commodities. However, it may also be useful as an oligopoly model of intangible items, such as securities, that are traded in a network of exchanges. In this case, our production and transmission constraints would represent some sort of financial or purchase constraints that limits the number of securities that buyers and arbitrageurs, respectively, can buy in exchanges.[11] In parallel to our paper, Malamud and Rostek (2013) develop a related model of decentralized exchanges. A difference in their work is that they do not consider production nor transport constraints, which are important for competition in electricity markets and other network industries, and they solve for linear asymmetric SFE with private information, similar to Kyle (1989) and Vives (2011), while we solve for non-linear symmetric SFE, similar to Klemperer and Meyer (1989).

2 The model

We shall consider markets for a single commodity that is traded over a network consisting of M nodes (markets) that are connected by N directed transport arcs (lines). We assume that each pair of nodes are connected by at most one arc. The network is *connected*, so that there is at least one path (chain of arcs) between every two nodes in the network. Thus we have that $N \geq M-1$. As is standard in graph theory, the topology of the network can be described by a *node-arc incidence* matrix $\mathbf{A}$ [11].[12] This matrix $\mathbf{A}$ has a row for every node and a column for every arc, and $ik-$th element a_{ik} defined as follows[13]:

$$a_{ik} = \begin{cases} -1, & \text{if arc } k \text{ is oriented away from node } i, \\ 1, & \text{if arc } k \text{ is oriented towards node } i, \\ 0, & \text{otherwise.} \end{cases}$$

Every arc starts in one node and ends in another node, so by definition we have that the rows of $\mathbf{A}$ add up to a row vector with zeros. Thus the rows are linearly dependent. It can be shown that the incidence matrix $\mathbf{A}$ of a connected network has rank $M-1$ [11].

The transported quantity in arc k is represented by the variable t_k which can be positive or negative, the latter indicating a flow in the opposite direction from the orientation of the arc. Thus the ith row of $\mathbf{At}$ represents the flow of the commodity into node i from the rest of the network. Transportation is assumed to be lossless and costless, but each arc k has a capacity K_k, so the vector $\mathbf{t}$ of arc flows satisfies

$$-\mathbf{K} \leq \mathbf{t} \leq \mathbf{K}. \tag{1}$$

[11] Some auctions, such as security auctions by the US treasury, use purchase constraints to prevent buyers from cornering the market.

[12] This is different to Wilson [45] who describes the network with *power transfer distribution factors* (PTDFs).

[13] Many authors adopt a different convention in which $a_{ik} = 1$ if arc k is oriented away from node i.

At each node i there are n_i suppliers who play a simultaneous move, one shot game. Each supplier offers a nondecreasing differentiable supply function

$$Q_{ig}(p),\ g = 1, 2, \ldots, n_i,$$

that defines how much each firm is prepared to supply at price p. For simplicity we assume that each firm is only active in one node. Non-strategic net-demand at each node i is $D_i(p) + \varepsilon_i$,[14] where $D_i(p)$ is a nonincreasing function of p and ε_i is a random local shock having a known probability distribution with joint density $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$. The demand shocks are realized after firms have committed to their supply functions. We denote the total deterministic net-supply in each node by $S_i(p_i) = \sum_{g=1}^{n_i} Q_{ig}(p_i) - D_i(p_i)$ and the vector with such components by $\mathbf{s}(\mathbf{p})$. We also introduce $S_{i,-g}(p_i) = \sum_{h=1,h\neq g}^{n_i} Q_{ih}(p_i) - D_i(p_i)$, which excludes the supply of firm g from the deterministic net-supply in node i.

We assume that the commodity is traded at the local market price of each node. In electric power networks this is called nodal pricing or locational marginal pricing (LMP) [15][23][39]. There is no storage in the nodes. Thus net-imports to each node must be equal to net-consumption in each node (consumption net of production). Hence, for each realization $\boldsymbol{\varepsilon}$ the market will be cleared by a set of prices that defines how much each supplier produces, how much is consumed in every node and what is transported through the network.

$$\mathbf{At}(\mathbf{p}) = \boldsymbol{\varepsilon} - \mathbf{s}(\mathbf{p}). \qquad (2)$$

There are many small price-taking arbitrageurs active in the network or one regulated, price-taking network operator. After the demand shocks have been realized, the arbitrageurs buy in some nodes, transport the commodity through the network without violating its physical constraints, and then sell it in other nodes. The market is cleared when all profitable feasible arbitrage trades have been exhausted.

Consumers and arbitrageurs are all price-takers. Thus if producers bid their true marginal costs, market clearing would lead to a competitive social welfare maximizing outcome. Arbitrage opportunities and the resulting clearing are the same for given supply function bids from producers, irrespective of their true production costs. Thus one way to compute the market clearing that the price-taking arbitrageurs give rise to is to solve for the social welfare maximizing outcome that would occur if submitted supply function bids would represent true marginal costs. We say that price-taking arbitrageurs lead to an outcome that maximizes stated social welfare. This is often how electricity markets with locational marginal pricing are cleared in practice and in theoretical models of such markets [15][17][18][26][45].

We solve for a Nash equilibrium of supply function bids, also called Supply Function Equilibrium (SFE). In this derivation, we assume that each producer is risk-neutral and chooses its supply function in order to maximize its expected profit. Ex-post, after demand shocks have been realized and prices and firms'

[14]Note that this is net-demand, so it is not necessarily non-negative. For example, fluctuating wind-power from small non-strategic firms can result in negative net-demand shocks.

outputs have been determined, the profit of firm g in node i is given by:

$$\Pi_{ig}(p,q) = pq - C_{ig}(q), \tag{3}$$

where p is the local price in node i, q is the output of the firm and $C_{ig}(q)$ is the firm's production cost, which is differentiable, convex and increasing up to its capacity constraint $\overline{q}_{ig}$.We let $\overline{p} > C'_{ig}(\overline{q}_{ig})$ be a reservation price.[15]

The residual demand curve of a firm is the market demand that is not met by other firms in the industry at a given price. The slope of this curve is important in the calculation of a firm's optimal supply function. The demand shocks are additive, so they will not change the slope of a firm's residual demand, as long as the same set of arcs are congested in the cleared market. Thus similar to Wilson [45] we find it useful to group shock outcomes for which the same set of arcs are congested in the cleared market. If two market outcomes for different $\boldsymbol{\varepsilon}$ realizations have exactly the same arcs with $t_k = -K_k$ and the same arcs with $t_k = K_k$ then we say that they are in the same congestion state ω. For each congestion state, we denote by $L(\omega)$, $B(\omega)$, and $U(\omega)$ the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively.

2.1 Optimality conditions

As in Anderson and Philpott [4], we use the market distribution function $\psi_{ig}(p,q)$ to characterize the uncertainty in the residual demand curve of firm g in node i. For given supply functions of the competitors this function returns the probability that the realized residual demand curve of firm g is to the left of the point (p,q).[16] We also refer to this as the rejection probability of the point offer (p,q) for firm g in node i. The market distribution function can be used to calculate the expected pay-off from the following line-integral [4]:

$$\int_{Q_{ig}(p)} \Pi_{ig}(p,q)\,d\psi_{ig}(p,q). \tag{4}$$

Thus, for any offer curve $Q_{ig}(p)$, the market distribution function contains all information of the residual demand that a firm needs to calculate its expected profit.[17] We define

$$Z(p,q) = \frac{\partial \Pi_{ig}}{\partial q}\frac{\partial \psi_{ig}}{\partial p} - \frac{\partial \psi_{ig}}{\partial q}\frac{\partial \Pi_{ig}}{\partial p} = \left(p - C'_{ig}(q)\right)\frac{\partial \psi_{ig}}{\partial p} - q\frac{\partial \psi_{ig}}{\partial q}. \tag{5}$$

[15] The reservation price is the highest price that the auctioneer is willing to pay for the commodity. Most auctions have reservation prices.

[16] Note that the market distribution function is analogous to Wilson's [46] probability distribution of the market price, which returns acceptance probabilities for offers. The main contribution of Anderson and Philpott's analysis is that it provides a global second-order condition for optimality.

[17] Anderson and Philpott derived the optimality condition for a firm in a single node. Their analysis did not consider network effects. However, it does not matter whether the rejection probability is driven by properties of the demand, competitors' supply functions or properties of the network. The optimality condition can be applied as long as the firm's accepted production is paid a (local) marginal price.

It can be shown that a supply function $Q_{ig}(p)$ is globally optimal if it satisfies [4]:

$$\begin{cases} Z(p,q) \geq 0 & \text{if } q < Q_{ig}(p) \\ Z(p,q) = 0 & \text{if } q = Q_{ig}(p) \\ Z(p,q) \leq 0 & \text{if } q > Q_{ig}(p). \end{cases} \quad (6)$$

In addition it is necessary for a local optimum that these conditions are locally satisfied at $q = Q_{ig}(p)$.

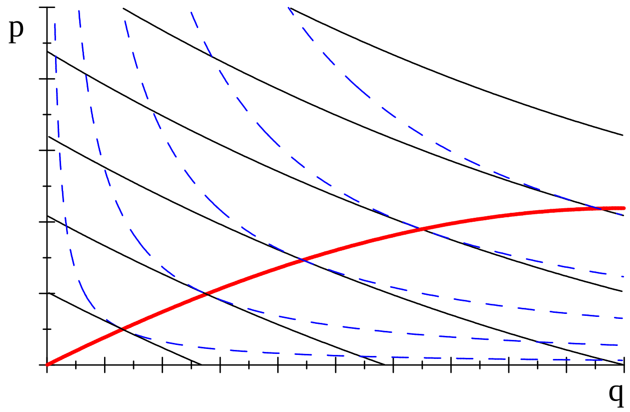

Figure 1: Example where contours of $\psi_{ig}(q,p)$ (thin) are linear. Isoprofit lines for $\pi_{ig}(q,p)$ are dashed. The optimal curve $q = Q_{ig}(p)$ (solid) passes through the points where these curves have the same slope.

In the simple case when the randomization is such that possible residual demand curves do not cross each other, then contours of $\psi_{ig}(p,q)$ correspond to possible realizations of the residual demand curves. In this case, the profit of firm g in node i is globally maximized if the payoff is globally optimized for each contour of $\psi_{ig}(p,q)$, i.e. the optimal supply function is ex-post optimal as in Klemperer and Meyer's single node model [29]. As illustrated in Figure 1, a necessary condition for this is that the supply function $Q_{ig}(p)$ crosses each contour of $\psi_{ig}(p,q)$ at a point where the latter is tangent to the firm's isoprofit line. Supply functions are no longer ex-post optimal when the market distribution function is generated by a randomization over crossing residual demand curves, but Anderson and Philpott show that the necessary condition still holds. The explanation is that as long as the market distribution function is the same, expected profits and the optimal offer for firm g do not change according to (4).

3 Radial networks

We begin our analysis by focusing on radial networks (i.e. trees with M nodes and $N = M - 1$ arcs forming an acyclic connected graph). The generalization to meshed networks with $N > M - 1$ is presented in Section 4. In radial networks there is a unique path (chain of arcs) between any two nodes in the network. Thus network flows are straightforwardly determined by net-supply in the nodes, which simplifies the clearing process of the market. We define $\boldsymbol{\rho}$ to be the vector of non-negative shadow prices (one for each arc) for flows in the positive direction. Similarly, we define $\boldsymbol{\sigma}$ to be the vector of non-negative shadow prices (one for each arc) for flows in the negative direction. Hence, the market clearing conditions for a radial network are[18]

$$\begin{array}{l} \mathbf{A}^{\top}\mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma} \\ 0 \leq \boldsymbol{\rho} \perp \mathbf{K} - \mathbf{t} \geq \mathbf{0} \\ 0 \leq \boldsymbol{\sigma} \perp \mathbf{K} + \mathbf{t} \geq \mathbf{0} \\ \mathbf{A}\mathbf{t}(\mathbf{p}) + \mathbf{s}(\mathbf{p}) = \boldsymbol{\varepsilon}. \end{array} \tag{7}$$

The first condition states that the shadow price for the arc gives the difference in nodal prices between its endpoints. The second and third set of conditions are called complementary slackness. They ensure that there are no feasible profitable arbitrage trades in the radial network. If two nodes are connected by a congested arc then the price at the importing end will be at least as large as the price in the exporting end. Another implication of the complementary slackness conditions is that nodes connected by uncongested arcs will form a zone with the same market price. We say that such nodes are completely integrated. The fourth condition ensures that net-demand equals net-imports in every node.

Recall that for a given congestion state ω, $L(\omega)$, $B(\omega)$, and $U(\omega)$ are the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively. Thus the complementary slackness conditions can be equivalently written as follows:

$$\begin{array}{llll} t_k = K_k, & \sigma_k = 0, & \rho_k \geq 0, & k \in U(\omega), \\ t_k \in (-K_k, K_k) & \sigma_k = 0, & \rho_k = 0, & k \in B(\omega), \\ t_k = -K_k, & \sigma_k \geq 0, & \rho_k = 0, & k \in L(\omega). \end{array}$$

Observe that given a congestion state ω and arc k, there is at most one variable t_k, ρ_k or σ_k that is not at a bound.

Recall that $\mathbf{A}$ has rank $M-1$ for radial networks, so the vector of prices $\mathbf{p}$ cannot be uniquely determined from the first market clearing condition in (7) by the choice of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. As in Wilson [45], we choose an arbitrary node i to be a *trading hub* with nodal price p. Let $\mathbf{1}_{M-1}$ be a column vector of $M-1$ ones and $\mathbf{0}_{M-1}$ be a column vector of $M-1$ zeros. Let $\mathbf{A}_i$ be row i of matrix $\mathbf{A}$, and let $\mathbf{A}_{-i}$ be matrix $\mathbf{A}$ with row i eliminated. For connected radial networks, it can be shown that $\mathbf{A}_{-i}$ is non-singular with determinant +1 or -1 [10]. Thus we can introduce

$$\mathbf{E} = \left(\left(\mathbf{A}_{-i} \right)^{T} \right)^{-1}. \tag{8}$$

We partition $\mathbf{t}$, $\mathbf{A}$, $\mathbf{E}$ and the shadow prices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ into ($\mathbf{t}_L$, $\mathbf{t}_B$, $\mathbf{t}_U$), ($\mathbf{A}_L$, $\mathbf{A}_B$, $\mathbf{A}_U$), ($\mathbf{E}_L$, $\mathbf{E}_B$, $\mathbf{E}_U$), ($\boldsymbol{\sigma}_L$, $\mathbf{0}_B$, $\mathbf{0}_U$) and ($\mathbf{0}_L$, $\mathbf{0}_B$, $\boldsymbol{\rho}_U$) corresponding to flows at their lower bounds, strictly between their bounds, and at their upper bounds.

[18] Section 4 provides a generalized and more formal derivation of these Karush-Kuhn-Tucker conditions.

Lemma 1 *In a radial network nodal prices $\mathbf{p}_{-i}$ can be expressed in terms of the price of the trading hub, p, and the shadow prices.*

$$\mathbf{p}_{-i} = p\mathbf{1}_{M-1} + \mathbf{E}_{U(\omega)}\boldsymbol{\rho}_{U(\omega)} - \mathbf{E}_{L(\omega)}\boldsymbol{\sigma}_{L(\omega)}. \tag{9}$$

Proof. We know that the columns of $\mathbf{A}^T$ sum to a column vector of zeros. Hence,

$$\begin{aligned}(\mathbf{A}_{-i})^T \mathbf{1}_{M-1} + \mathbf{A}_i^T &= \mathbf{0}_{M-1}\\ \left((\mathbf{A}_{-i})^T\right)^{-1} \mathbf{A}_i^T &= -\mathbf{1}_{M-1}.\end{aligned}$$

Using this result, we can write the market clearing condition $\mathbf{A}^\top\mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma}$ as follows:

$$\begin{aligned}(\mathbf{A}_{-i})^T \mathbf{p}_{-i} + p\mathbf{A}_i^T &= \boldsymbol{\rho} - \boldsymbol{\sigma}\\ \mathbf{p}_{-i} &= \left((\mathbf{A}_{-i})^T\right)^{-1} \left(\boldsymbol{\rho} - \boldsymbol{\sigma} - p\mathbf{A}_i^T\right)\\ \mathbf{p}_{-i} &= p\mathbf{1}_{M-1} + \left((\mathbf{A}_{-i})^T\right)^{-1} \left(\boldsymbol{\rho} - \boldsymbol{\sigma}\right).\end{aligned} \tag{10}$$

(8) now gives the stated result. ■

For any index set Υ of columns of $\mathbf{A}$ (or equivalently any set Υ of arcs) we will find it useful to define the volume that feasible flows and shadow prices associated with arcs in Υ can span. Thus we define the sets

$$\begin{aligned}\mathcal{T}(\Upsilon_1) &= \{\mathbf{t}_{\boldsymbol{\Upsilon}_1} \mid -\mathbf{K}_{\Upsilon_1} \leq \mathbf{t}_{\boldsymbol{\Upsilon}_1} \leq \mathbf{K}_{\Upsilon_1}\},\\ \mathcal{U}(\Upsilon_2) &= \{\boldsymbol{\rho}_{\boldsymbol{\Upsilon}_2} \mid \mathbf{0} \leq \boldsymbol{\rho}_{\boldsymbol{\Upsilon}_2}\},\\ \mathcal{L}(\Upsilon_3) &= \{\boldsymbol{\sigma}_{\boldsymbol{\Upsilon}_3} \mid \mathbf{0} \leq \boldsymbol{\sigma}_{\boldsymbol{\Upsilon}_3}\}.\end{aligned} \tag{11}$$

$\mathcal{T}(\Upsilon_1)$ is the volume in $\mathbf{t}$ space that the flows in a set of uncongested arcs Υ_1 can span. $\mathcal{U}(\Upsilon_2)$ and $\mathcal{L}(\Upsilon_3)$ are the volumes in $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space spanned by the shadow prices of congested arcs in the sets Υ_2 and Υ_3, respectively. In particular we are interested in $\mathcal{S}(\omega) \subseteq \mathbb{R}^{M-1}$, which we define by

$$\mathcal{S}(\omega) = \mathcal{L}(L(\omega)) \times \mathcal{U}(U(\omega)) \times \mathcal{T}(B(\omega)). \tag{12}$$

Hence, $\mathcal{S}(\omega)$ is the total volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space that is spanned for a congestion state ω.

3.1 Optimality conditions for radial networks

In the following we take supply functions $Q_{jh}(p)$ of the competitors as given and we want to calculate the best response of firm i in node g. For notational convenience we let node i, the node under study, be the trading hub with price p.[19] We denote by $\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma})$ the vector of nodal prices defined by (9), where we choose to suppress the dependence on ω for notational convenience. We define $\mathbf{s}(\mathbf{p}, q)$ to be the vector of nodal net-supply functions with jth component

$$\begin{cases} q + \sum_{h=1,h\neq g}^{n_i} S_{ih}(p) - D_i(p), & j = i,\\ \sum_{h=1}^{n_j} S_{jh}(p_j) - D_j(p_j). & j \neq i.\end{cases} \tag{13}$$

[19] The trading hub is moved and a new price relation as in (9) is calculated when optimality conditions are derived for a firm in another node.

For each state ω we partition the nodes into the sets $\Xi(\omega)$ and $\digamma(\omega)$. $\Xi(\omega)$ includes all nodes that are completely integrated with node i (the trading hub), where firm g is located, through some uncongested chain of arcs. The set $\digamma(\omega)$ contains all other nodes in the network. Similarly we partition the shock vector into $\boldsymbol{\varepsilon}_{\Xi(\omega)}$ and $\boldsymbol{\varepsilon}_{\digamma(\omega)}$. The arcs are partitioned as follows. We let $\mathbf{t}_{\Xi(\omega)}$ be the flows in the uncongested arcs between nodes in the set $\Xi(\omega)$ and we let $\mathbf{t}_{\digamma(\omega)}$ be the vector of flows in the other arcs. In particular, the vector $\mathbf{t}^{\digamma}_{B(\omega)}$ denotes uncongested flows in the other arcs. $M_{\Xi(\omega)}$ is the number of nodes in $\Xi(\omega)$ and we note that they must be connected by $M_{\Xi(\omega)}-1$ uncongested arcs. We use the node-arc incidence matrix $\mathbf{A}_{\Xi(\omega)}$ to describe the connected radial network with nodes in $\Xi(\omega)$ and arcs with uncongested flows $\mathbf{t}_{\Xi(\omega)}$ connecting nodes in this set. We let $\mathbf{A}_{\digamma(\omega)}$ be a node-arc incidence matrix with $M-M_{\Xi(\omega)}$ rows/nodes and $M-M_{\Xi(\omega)}$ columns/arcs, describing the rest of the network. [20]

Proposition 1 *In a radial network, the optimal output q of firm g at price p in node i can be determined from the following Z function:*

$$Z=\left(p-C'_{ig}(q)\right)\sum_{\omega}\left(S'_{i,-g}(p)+\sum_{k\in\Xi(\omega)\backslash i}S'_k(p)\right)P(p,q,\omega)-q\sum_{\omega}P(p,q,\omega) \tag{14}$$

where

$$P(p,q,\omega)=\int_{\mathcal{S}(\omega)} f(\mathbf{At}+\mathbf{s}(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),q))\,J_{\digamma}(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)} \tag{15}$$

$$J_{\digamma}(\omega)=\left|\frac{\partial\boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial(\mathbf{t}^{\digamma}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)})}\right|. \tag{16}$$

Row k of the Jacobian matrix $\frac{\partial\boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial\left(\mathbf{t}^{\digamma}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)}$ can be constructed as follows for the state ω:

$$\left(\frac{\partial\boldsymbol{\varepsilon}_{\digamma(\omega)}}{\partial\left(\mathbf{t}^{\digamma}_{B(\omega)},\boldsymbol{\rho}_{U(\omega)},\boldsymbol{\sigma}_{L(\omega)}\right)}\right)_k=\left[\begin{array}{ccc}\left(\mathbf{A}^{\digamma}_{B(\omega)}\right)_k & S'_k(p_k)\left(\mathbf{E}_{U(\omega)}\right)_k & -S'_k(p_k)\left(\mathbf{E}_{L(\omega)}\right)_k\end{array}\right] \tag{17}$$

for $k\in\digamma(\omega)$.

Proof. In order to apply the optimality conditions in (5) and (6), we need to derive the market distribution function $\psi_{ig}(p,q)$. It is the probability that firm g in node i sells less than q units if the market price in node i is p. Thus the calculation of this function involves determining a market outcome for every realization of the vector $\boldsymbol{\varepsilon}$, and then integrating the density function f over the volume in ε-space that corresponds to firm g's point offer (p,q) being rejected. In the general case this volume is complicated and it is even more complicated to

[20] Note that the remainder of the network has at least one arc that is lacking its start or end node. Also the remainder of the network is not necessarily connected.

differentiate $\psi_{ig}(p,q)$ (we need such derivatives in our optimality conditions) if one follows this direct approach. Like Wilson [45], we avoid this by transforming the problem into one where we instead integrate over the flows and shadow prices that arise in each congestion state. When calculating $\frac{\partial \psi_{i,g}(p,q)}{\partial p}$ we keep the output of firm g fixed while the price p at node i is free to change. Thus we calculate $\psi_{ig}(p,q)$ from the probability that the price in node i, π, is below p when firm g's offer is fixed to q. We want to transform the volume in $\boldsymbol{\varepsilon}-$space into a corresponding volume in $\mathbf{t}$, $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$ and π space for variables that are not at a bound. To make this substitution of variables when computing the multi-dimensional integral, we need the following factor to represent the change in measure [8, p. 368].

$$J_p(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial \left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \pi\right)} \right|, \tag{18}$$

the absolute value of the determinant of the Jacobian matrix representing the change of variables. Thus the probability that the market clearing price π at node i is less than p can be calculated from

$$\psi_{i,g}(p,q) = \sum_{\omega} \int_{\pi=-\infty}^{p} \int_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}(\mathbf{p}(\pi, \boldsymbol{\rho}, \boldsymbol{\sigma}), q)\right) J_p(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)} d\pi, \tag{19}$$

where $\mathcal{S}(\omega)$ is defined in (12). It is now straightforward to show that:

$$\frac{\partial \psi_{i,g}(p,q)}{\partial p} = \sum_{\omega} \int_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right)\right) J_p(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)}. \tag{20}$$

When calculating $\frac{\partial \psi_{i,g}(p,q)}{\partial q}$ we keep p fixed in node i, while q, the output of firm g, is free to change. Thus we calculate $\psi_{ig}(p,q)$ from the probability that the firm's realized output, r, is q or lower when the price in node i is fixed to p. In this case, the substitution factor is given by:

$$J_q(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}}{\partial \left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, r\right)} \right|. \tag{21}$$

The probability that the market clearing quantity r for generator g at node i is less than q can now be calculated from

$$\psi_{i,g}(p,q) = \sum_{\omega} \int_{r=-\infty}^{q} \int_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), r\right)\right) J_q(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)} dr \tag{22}$$

so

$$\frac{\partial \psi_{i,g}(p,q)}{\partial q} = \sum_{\omega} \int_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right)\right) J_q(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)}. \tag{23}$$

The next step is to calculate the Z function of firm g in node i. We substitute results in Lemma 7 and Lemma 8 (see Appendix A) into (20) and (23). Next, we get (14) and (15) by substituting (20) and (23) into (5). $J_F(\omega)$ can be determined from Lemma 9 in the Appendix. ■

Observe that $P(p,q,\omega)$ is a probability density in the sense that $P(p,q,\omega)\,dq$ is the probability that firm g is dispatched in the interval $(q, q+dq)$ and the congestion state is ω given that the clearing price is p. The term $\sum_\omega P(p,q,\omega)$ is then the probability that g is dispatched in the interval $(q, q+dq)$ given that the clearing price is p. The first-order optimality condition is given by $Z(p,q) = 0$, so it immediately follows from (14) that:

Corollary 1 *The optimal output q of firm g in node i at price p satisfies the first-order condition:*

$$q = \left(p - C'_{ig}(q)\right) \sum_{\omega} \left(S'_{i,-g}(p) + \sum_{k \in \Xi(\omega) \setminus i} S'_k(p) \right) P(\omega | p, q), \tag{24}$$

where

$$P(\varpi | p, q) := \frac{P(p,q,\varpi)}{\sum_\omega P(p,q,\omega)}$$

is the conditional probability that the network is in state ϖ given that the price in node i is p and firm g has output q.

In a single-node network, the optimal output of a producer is proportional to its mark-up and the slope of the residual demand that it is facing [29]. In a network with multiple connected nodes, producer g in node i only faces the slope of the net-supply in nodes that are completely integrated with its own node. Thus according to Corollary 1, the slope of net-supply in each other node is scaled by the conditional probability that this node is completely integrated with node i. Hence, for multi-dimensional shocks, Klemperer and Meyer's condition generalizes to saying that the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand that it is facing. This first-order condition is consistent with Wilson's results [45]. We notice that in case all arcs have unlimited capacities, we get the Klemperer and Meyer condition for a completely integrated network. The other extreme when all arcs have zero capacity, yields the Klemperer-Meyer equation for the isolated node i.

Definition 1 *For firm g in node i we define the market integration function by*

$$\mu_{ig}(p,q) = \sum_{\omega} \sum_{k \in \Xi(\omega)} P(\omega|p,q) = \sum_{\omega} M_{\Xi(\omega)} P(\omega|p,q).$$

Thus, the market integration function is equal to the expected number of nodes (including node i itself) that are completely integrated with node i given that firm g has output q and node i has the market price p.

Lemma 2 *In a symmetric equilibrium where each node has demand $D(p)$ and n producers each submitting a supply function $Q(p)$, the first-order condition can be written:*

$$Q = (p - C'(Q))\left((\mu(p,Q)\,n - 1)\,Q' - \mu(p,Q)\,D'\right). \tag{25}$$

Proof. We first substitute $S_k(p) = nQ(p) - D(p)$ and $S_{i,-g}(p) = (n-1)Q(p) - D(p)$ into (24).

$$Q = \left(p - C'_{ig}(Q)\right) \sum_{\omega} \left((n-1)Q' - D' + \sum_{k \in \Xi(\omega) \setminus i} (nQ' - D') \right) P(\omega | p, Q).$$

We have $\sum_{\omega} P(\omega|p,q) = 1$ and by definition $i \in \Xi(\omega)$, so it follows from Definition 1 that $\sum_{\omega} \sum_{k \in \Xi(\omega) \setminus i} P(\omega|p,q) = \mu(p,Q) - 1$. Thus

$$Q = \left(p - C'_{ig}(Q)\right)\left((n-1)Q' - D' + (\mu(p,Q) - 1)(nQ' - D')\right),$$

which gives (25). ∎

3.2 Examples

By means of Corollary 1 we are able to construct a first-order condition for each firm in a radial network. The supply function equilibrium (SFE) can be solved from a system of such first-order conditions for general radial networks. The global second-order condition of an available first-order solution can be verified by (6). In this section we use these optimality conditions to derive SFE for two-node and star networks with symmetric firms.

3.2.1 Two node network

Consider a simple network with two nodes connected by one arc from node 1 to node 2 with flow $t \in [-K, K]$. We derive the optimality condition for a firm in node 1, and thus we pick node 1 as being the trading hub with price p. It can be shown that:

Lemma 3 *In a two-node network, firm g's optimality condition in node* 1 *is given by:*

$$\begin{array}{c} Z(p,q) = (p - C'_{1g}(q))(S'_{1,-g}(p) + S'_2(p))P(p,q,\omega_1) \\ +(p - C'_{1g}(q))S'_{1,-g}(p)\left(P(p,q,\omega_2) + P(p,q,\omega_3)\right) \\ -q(P(p,q,\omega_1) + P(p,q,\omega_2) + P(p,q,\omega_3)) = 0, \end{array} \tag{26}$$

where

$$\begin{array}{c} P(p,q,\omega_1) = \int_{-K}^{K} f(q + S_{1,-g}(p) - t, S_2(p) + t)\, dt \\ P(p,q,\omega_2) = \int_{S_2(p)+K}^{\infty} f(q + S_{1,-g}(p) - K, \varepsilon_2)\, d\varepsilon_2 \\ P(p,q,\omega_3) = \int_{-\infty}^{S_2(p)-K} f(q + S_{1,-g}(p) + K, \varepsilon_2)\, d\varepsilon_2. \end{array} \tag{27}$$

Proof. See Appendix B ∎

Figure 2 gives a geometric view of the probabilities in (27) for the special case when firm g is the only producer in node 1, so that $S_{1,-g}(p) = 0$.

Below we consider symmetric NE for symmetric firms and symmetric shock densities. The existence of an equilibrium depends on the partial derivatives $f_i(\varepsilon_1, \varepsilon_2)$, $i = 1, 2$, of the shock density which must be sufficiently small. It can be shown that symmetric solutions to (25) are equilibria under the following circumstances.

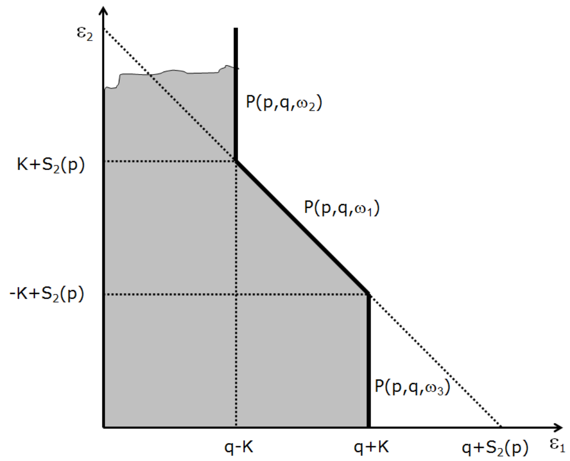

Figure 2: Computation of $P(p, q, \omega)$ when firm g is the only producer in node 1. The probability mass in the shaded area is equal to $\psi_{1,g}(q, p)$, the probability that the residual demand curve of firm g is to the left of the point (p, q). The values of $P(p, q, \omega)$ are integrals along the right-hand boundary of this region as shown.

Proposition 2 *Consider a two-node network with n symmetric firms in each node, each firm having identical production capacities $\overline{q}$ and identical marginal costs that are either constant or strictly increasing. If demand is inelastic up to a reservation price $\overline{p} > C'(\overline{q})$, and has a bounded shock density that satisfies $f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_2, \varepsilon_1) > 0$ and $2n\overline{q}\,|f_i(\varepsilon_1, \varepsilon_2)| \leq (3n-2)\, f(\varepsilon e_1, \varepsilon_2)$ when $(\varepsilon_1, \varepsilon_2) \in [-K, \overline{q}+K] \times [-K, \overline{q}+K] : \{\varepsilon_1 + \varepsilon_2 \leq 2\overline{q}\}$, then there exists a unique symmetric supply function equilibrium in the network, where each firm's monotonic offer, $Q(p)$, can be calculated from:*

$$Q'(p) = \frac{Q}{(p - C'(Q))\,(n\mu(nQ) - 1)} \tag{28}$$

$$\mu(nQ) = 1 + \frac{P(nQ, \omega_1)}{\sum_\omega P(nQ, \omega)} \tag{29}$$

for $p \in (C'(0), \overline{p}]$ with the initial condition $Q(\overline{p}) = \overline{q}$. The probabilities $P(nQ, \omega)$ are given by

$$\begin{aligned} P(nQ, \omega_1) &= \int_{-K}^{K} f(nQ - t, nQ + t)\, dt \\ P(nQ, \omega_2) &= \int_{nQ+K}^{\infty} f(nQ - K, \varepsilon_2)\, d\varepsilon_2 \\ P(nQ, \omega_3) &= \int_{-\infty}^{nQ-K} f(nQ + K, \varepsilon_2)\, d\varepsilon_2. \end{aligned} \tag{30}$$

Proof. See Appendix B ■

Symmetric offers $Q(p)$ depend on the number of firms per node and their production costs. Still, for inelastic demand and symmetric equilibria, it can be noted from (29) that the market integration function $\mu(nQ)$ only depends on the total nodal output and exogenous parameters: the demand shock distribution and production and transport capacities. The market integration function does not depend on market competitiveness or firms' production costs. The reason is that consumers are in this case insensitive to the price, so the number of production units that are needed to meet a given demand shock outcome does not depend on market competition nor production costs. Moreover, the order in which production units of symmetric firms are accepted is the same irrespective of their symmetric mark-ups. It follows from (28) that oligopoly producers will increase their mark-ups at output levels where the market integration function $\mu(nQ)$ is small, i.e. when the arc is congested with a high conditional probability. Similarly, oligopoly producers will decrease their mark-ups at output levels where the market integration function is large.

In the next step we will explicitly solve for symmetric SFE in the two-node network. To simplify the optimality conditions we consider the case where demand shocks follow a bivariate uniform distribution.

Assumption 1: *Consider a network with two nodes connected by an arc with capacity K* and *with n symmetric firms in each node. Inelastic demand in each node is given by ε_i. We assume that shocks are uniformly distributed with a constant density, $\frac{1}{V_1}$, on the surface $(\varepsilon_1, \varepsilon_2) \in [-K, n\overline{q}+K] \times [-K, n\overline{q}+K]$: $\{0 \leq \varepsilon_1 + \varepsilon_2 \leq 2n\overline{q}\}$ and zero elsewhere.*

Proposition 3 *Make Assumption 1, then the symmetric market integration function for a two node network is given by*

$$\mu = 1 + P(\omega_1|p, q) = \frac{4K + n\overline{q}}{2K + n\overline{q}}. \tag{31}$$

Solutions to (25) are SFE, and the inverse symmetric supply functions can be calculated from:

$$p(Q) = Q^{-1}(Q) = \frac{\overline{p}Q^{\mu n-1}}{\overline{q}^{\mu n-1}} + (\mu n - 1) Q^{\mu n-1} \int_Q^{\overline{q}} \frac{C'(u)\, du}{u^{\mu n}}. \tag{32}$$

Proof. See Appendix B. ■

It follows from Proposition 3 that the market integration function μ simplifies to a constant for uniformly distributed demand shocks. In this case, the equilibrium offer of a firm in the two-node network with n symmetric firms per node is identical to the equilibrium offer of a firm in an isolated node with μn symmetric firms. Fig. 3 illustrates how the total supply function in a node depends on μn if the total production capacity in each node is kept fixed. As the equations are identical, the symmetric SFE of the network also inherits the following properties from the single node case [24][25].

Corollary 2 *Solutions to (32) have the following properties*

1. Mark-ups are positive for a positive output.
2. For a given nodal production cost function, mark-ups decrease at every nodal output level with more firms in the market.

Proposition 2 ensures existence of equilibria when slopes in the shock density are sufficiently small. However, existence is problematic for steep slopes in the shock density and especially so when it has discontinuities. This is illustrated by the non-existence example below.

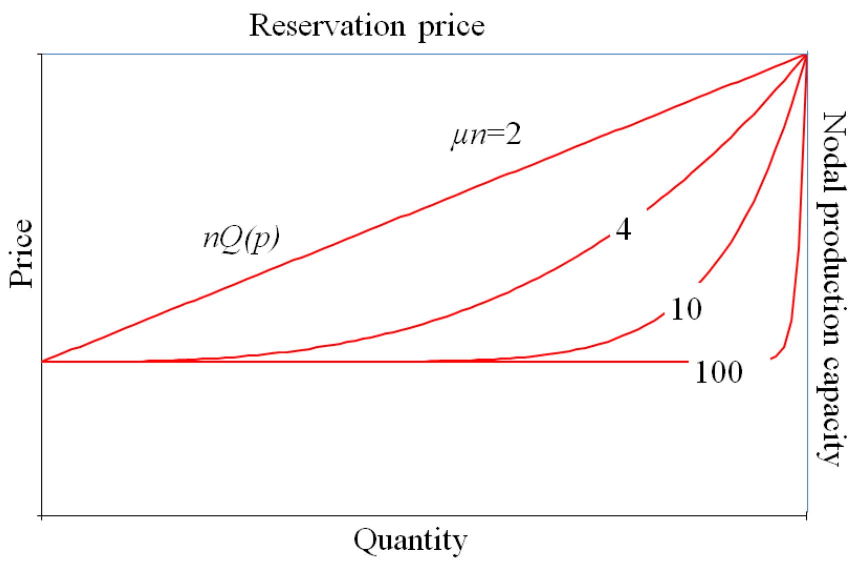

Figure 3: Nodal supply curve in one node with inelastic demand up to a reservation price and constant marginal costs up to a fixed nodal production capacity. The network is symmetric with n firms per node. Demand shocks are uniformly distributed so that the market integration function μ is constant.

Example 1 *Shock densities with discontinuities: Assume that the support of the shock ε_i, $i \in \{1,2\}$ is given by $[0,\overline{\varepsilon}]$. The density is differentiable inside the support set, but decreases discontinuously to zero when $\varepsilon_1 = \overline{\varepsilon}$ and $\varepsilon_2 \in [0,\overline{\varepsilon}]$, where*

$$2K < \overline{\varepsilon} < \overline{q} + K, \tag{33}$$

which would violate Assumption 1. Consider a potential symmetric NE of a duopoly market with one firm in each node with identical costs $C(q)$. Assume that the symmetric supply functions $Q(p)$ are monotonic, that demand is inelastic, so that $S_i(p) = Q(p)$. In the following we will show that firm 1 will have a profitable deviation from the potential symmetric pure-strategy NE. In particular we will consider the point (q_0, p_0) where

$$q_0 = Q(p_0) = \overline{\varepsilon} - K. \tag{34}$$

It follows from (33) that $q_0 \in (0, \overline{q})$. Thus, unlike the distribution in Assumption 1, the shock density can reach its discontinuity even if the transport capacity is non-binding. From (27) and symmetric supply functions we have:

$$P(p,q,\omega_3) = \int_{-\infty}^{Q(p)-K} f(q+K,\varepsilon)d\varepsilon$$

and accordingly

$$\lim_{q\uparrow\overline{\varepsilon}-K} P(p,q,\omega_3) > \lim_{q\downarrow\overline{\varepsilon}-K} P(p,q,\omega_3) = \lim_{q\downarrow\overline{\varepsilon}-K} \int_{-\infty}^{Q(p)-K} f(q+K,\varepsilon)d\varepsilon = 0. \tag{35}$$

However, $P(p,q,\omega_1)$ and $P(p,q,\omega_2)$ are still continuous at the point (q_0,p_0). From (27) we have:

$$P(p_0,q_0,\omega_1) = \int_{-K}^{K} f(q_0 - t, q_0 + t)dt = \int_{-K}^{K} f(\overline{\varepsilon} - K - t, \overline{\varepsilon} - K + t)dt > 0$$
$$P(p_0,q_0,\omega_2) = \int_{K+q_0}^{\infty} f(q_0 - K, \varepsilon)d\varepsilon = \int_{\overline{\varepsilon}}^{\infty} f(\overline{\varepsilon} - 2K, \varepsilon)d\varepsilon = 0$$

so (35) implies that

$$\lim_{q\downarrow\overline{\varepsilon}-K} \sum_{\omega} P(p_0,q,\omega) < \lim_{q\uparrow\overline{\varepsilon}-K} \sum_{\omega} P(p_0,q,\omega). \quad (36)$$

A necessary condition for the solution being an equilibrium is that the optimality condition in (6) is locally satisfied at the point $(p_0, \overline{\varepsilon} - K)$. Thus we must have $\lim_{q\uparrow\overline{\varepsilon}-K} Z(p_0,q) \geq 0$, but together with (26) and (36) this would imply that

$$\begin{aligned}\lim_{q\downarrow\overline{\varepsilon}-K} Z(p_0,q) &= (p - C'(q_0))Q'(p_0)P(p_0,q_0,\omega_1) - q_0(\lim_{q\downarrow\overline{\varepsilon}-K} \sum_{\omega} P(p_0,q,\omega)) \\ &> (p - C'(q_0))Q'(p_0)P(p_0,q_0,\omega_1) - q_0(\lim_{q\uparrow\overline{\varepsilon}-K} \sum_{\omega} P(p_0,q,\omega)) \\ &= 0,\end{aligned}$$

which would violate the local second-order condition in (6), and accordingly there is a profitable deviation from the symmetric solution $Q(p)$.

The next example illustrates that existence of SFE is problematic if shocks are perfectly correlated. Similar to the incentives to congest analysed by Borenstein et al. [14], correlated shocks give a producer in an importing node the incentive to unilaterally deviate from the first-order solution by withholding power in order to congest the transmission line so as to increase the price of the importing node.

Example 2 ***Perfectly correlated shocks:*** *Consider two nodes connected by one arc. Demand shocks in the two nodes are perfectly correlated. This means that market prices are driven by a one-dimensional uncertainty. We assume that the demand shocks in both nodes are strictly increasing with respect to this underlying one-dimensional shock.*[21] *We also assume that $D_i' < 0$, $i \in \{1,2\}$, so that $S_1'(p)$*

[21] In his analysis of perfectly correlated shocks, Wilson [45] focuses on the special case when the shock at node 1 is fixed to zero. This means that regardless of deviations in node 2, exports from node 1 can never congest the arc below the price p^*. Thus the profitable deviation that is outlined in our example does not exist in this special case. We have found that ex-post optimal SFE can be constructed for such special cases. For similar reasons we have found that SFE can be constructed when demand shocks in the two nodes are negatively correlated. However, these equilibria are more complicated as one of the nodal shocks will decrease with respect to the underlying shock. The price in this node will first increase with respect to the one-dimensional underlying shock until the arc is congested and then decrease with respect to the underlying shock. Thus such SFE are not ex-post optimal.

and $S_2'(p)$ are always strictly positive. Thus both nodal prices are strictly increasing in the underlying shock, and there is a one-to-one mapping between the underlying shock and each nodal price. In equilibrium, firms maximize their profits by choosing a supply function that optimizes the output for each price and shock, so that the equilibrium becomes ex-post optimal as in Klemperer and Meyer's [29] model of single markets. Without loss of generality, assume that the arc from node 1 to 2 is congested at the price p^* and uncongested in some range $(\widehat{p}, p^*)$. Assume that the first-order condition results in a well-behaved monotonic solution for each firm where mark-ups are strictly positive in the range $(\widehat{p}, p^*]$. Consider a firm g in node 2 (the importing node), with the first-order solution $Q_{2g}(p)$. We choose $\widehat{p}$ sufficiently close to p^* and assume that the shock density is well-behaved so that $P(p,q,\omega_1) + P(p,q,\omega_2) + P(p,q,\omega_3)$ is well-defined and bounded away from zero for $(p,q) \in (\widehat{p}, p^*) \times (Q_{2g}(\widehat{p}), Q_{2g}(p^*))$. To simplify the analysis we consider the case when firms have constant marginal costs. We use (26) and consider the ratio

$$\widehat{Z}_{2g}(p,q) := \frac{Z_{2g}(p,q)}{\sum_{\omega} P(p,q,\omega)} = (p - C_{2g}')(S_1'(p) + S_{2,-g}'(p))P(\omega_1|p,q) + (p - C_{2g}')S_{2,-g}'(p)\, P(\omega_2 \cup \omega_3|p,q) - q,$$

where $P(\omega_1|p,q) = \frac{P(p,q,\omega_1)}{P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)}$ is the conditional probability that the arc is uncongested and $P(\omega_2 \cup \omega_3|p,q) = \frac{P(p,q,\omega_3)+P(p,q,\omega_2)}{P(p,q,\omega_1)+P(p,q,\omega_2)+P(p,q,\omega_3)}$ is the conditional probability that the arc is congested. It follows from our assumptions that the first-order solution satisfies:

$$\begin{aligned} P(\omega_1|p, Q_{2g}(p)) &= \begin{cases} 1 & \text{if } p < p^* \\ 0 & \text{if } p \geq p^* \end{cases} \\ P(\omega_2 \cup \omega_3|p, Q_{2g}(p)) &= \begin{cases} 0 & \text{if } p < p^* \\ 1 & \text{if } p \geq p^* \end{cases} \end{aligned} \tag{37}$$

and that

$$Z_{2g}(p, Q_{2g}(p)) = \widehat{Z}_{2g}(p, Q_{2g}(p)) = 0. \tag{38}$$

Consider a price $p_0 \in (\widehat{p}, p^*)$. Since $S_1'(p) > 0$ and mark-ups are strictly positive for $p \in (\widehat{p}, p^*)$,

$$p_0 - C_{2g}' S_1'(p_0) \geq \inf_{p \in (\widehat{p}, p^*)} \{(p - C_{2g}')S_1'(p)\} = \Delta > 0.$$

We proceed to construct a deviation for the function $Q_{2g}(p)$ that improves the payoff of firm g. The shock at node 1 is increasing in the underlying one-dimensional shock, so for prices p_0 sufficiently close to p^* it is possible for firm g to withhold an amount of production $\delta_0 \in (0, \Delta)$ so that $P(\omega_1|p_0, Q_{2g}(p_0) - \delta_0) = 0$ and $P(\omega_2 \cup \omega_3|p_0, Q_{2g}(p_0) - \delta_0) = 1$. Let δ_1 be the infimum of such δ_0. This implies that for every $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$

$$\begin{aligned} \widehat{Z}_{2g}(p_0, Q_{2g}(p_0)) - \widehat{Z}_{2g}(p_0, Q_{2g}(p_0) - \delta) &= (p_0 - C_{2g}')(S_1'(p_0) + S_{2,-g}'(p_0)) - Q_{2g}(p_0) \\ &\quad -((p_0 - C_{2g}')S_{2,-g}'(p_0) - (Q_{2g}(p_0) - \delta)) \\ &= (p_0 - C_{2g}')S_1'(p_0) - \delta \\ &> \frac{\Delta - \delta_1}{2}. \end{aligned}$$

It follows from (38) that for every $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$, $\widehat{Z}_{2g}(p_0, Q_{2g}(p_0) - \delta) < -(\frac{\Delta-\delta_1}{2}) < 0$, *and so*

$$Z_{2g}(p_0, Q_{2g}(p_0) - \delta) < -h(\frac{\Delta - \delta_1}{2}) \tag{39}$$

for some constant $h > 0$, *where* h *is less than or equal to the infimum of* $P(p_0, Q_{2g}(p_0) - \delta, \omega_1) + P(p_0, Q_{2g}(p_0) - \delta, \omega_2) + P(p_0, Q_{2g}(p_0) - \delta, \omega_3)$ *over* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$. *Withholding less than* δ_1 *units at* p_0 *only has a second-order effect on* $\widehat{Z}_{2g}$ *and* Z_{2g}. *The deviation in* $Q_{2g}(p_0)$ *starts at* $p_\delta < p_0$, *which we define by*

$$Q_{2g}(p_\delta) = Q_{2g}(p_0) - \delta.$$

We assume that p_0 *is sufficiently close to* p^*, *so that we can find a sufficiently small* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$ *to ensure that* $p_\delta > \widehat{p}$. *For some* $\eta_1 > 0$, *when* $p \in (p_\delta + \eta_1, p_0)$, *the line is congested when the offer is* $Q_{2g}(p_\delta)$ *at price* p. *It follows from (38) and (37) that*

$$\begin{aligned}
\widehat{Z}_{2g}(p, Q_{2g}(p_\delta)) &= \widehat{Z}_{2g}(p, Q_{2g}(p_\delta)) - \widehat{Z}_{2g}(p, Q_{2g}(p)) \\
&= (p - C'_{2g})S'_{2,-g}(p) - Q_{2g}(p_\delta) \\
&\quad -((p - C'_{2g})(S'_1(p) + S'_{2,-g}(p)) - Q_{2g}(p)) \\
&= Q_{2g}(p) - Q_{2g}(p_0) + \delta - (p - C'_{2g})S'_1(p) \\
&< -(\Delta - \delta) < -(\frac{\Delta - \delta_1}{2})
\end{aligned}$$

for $p \in (p_\delta + \eta_1, p_0)$. *Thus*

$$Z_{2g}(p, Q_{2g}(p_\delta)) < -k(\frac{\Delta - \delta_1}{2})$$

for $p \in (p_\delta + \eta_1, p_0)$ *and some positive*

$$k \le \inf_{p \in (p_\delta + \eta_1, p_0)} \{P(p, Q_{2g}(p_\delta), \omega_1) + P(p, Q_{2g}(p_\delta), \omega_2) + P(p, Q_{2g}(p_\delta), \omega_3)\}.$$

Together with (39) this implies that if we integrate Z *along the deviation defined by* δ, *then*

$$\int_{p_\delta}^{p_0} Z_{2g}(p, Q_{2g}(p_\delta))dp + \int_{Q_{2g}(p_\delta)}^{Q_{2g}(p_0)} Z_{2g}(p_0, q)dq < 0, \tag{40}$$

if we choose p_0 *sufficiently close to* p^* *and* $\delta \in \left(\delta_1, \frac{\Delta+\delta_1}{2}\right)$ *sufficiently small, so that first-order effects dominate second-order effects. (40) violates a necessary local optimality condition [4]. The intuition is that a producer in an importing node always has an incentive to unilaterally deviate from the first-order solution by withholding power in order to congest the arc at lower prices than* p^*, *which increases the price of the importing node.*

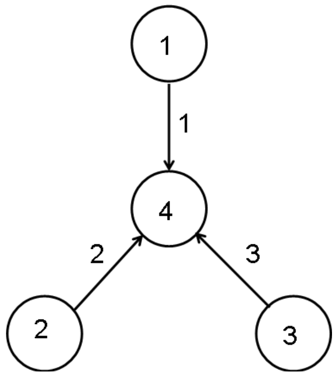

Figure 4: Star network example.

3.2.2 Star network

Next, we consider a star network with four nodes and three radial lines with capacity K, as shown in Figure 4. We define all arcs to be directed towards the center node 4. Each arc has the same number as the starting node, i.e. $1, 2$ or 3.

Demand shocks are defined on the following region Θ:

$$\Theta = \left\{ \begin{array}{c} (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4 | -K \le \varepsilon_i \le n\bar{q} + K, -3K \le \varepsilon_4 \le 3K, \\ 0 \le \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \le 3n\bar{q} \; \forall i \in \{1, 2, 3\}. \end{array} \right\}$$

and we let V_2 be the volume of this region.

Assumption 2. *Consider a star network with four nodes and three radial lines with capacity K directed towards the center node 4. There are n firms with identical costs $C(q)$ in each node $1 - 3$. There are no producers in node 4 (the center node). Inelastic demand in nodes $i \in \{1, 2, 3, 4\}$ is given by ε_i. Demand shocks are uniformly distributed such that:*

$$f(\boldsymbol{\varepsilon}) = \left\{ \begin{array}{cc} \frac{1}{V_2} & \text{if } \boldsymbol{\varepsilon} \in \Theta \\ 0 & \text{otherwise.} \end{array} \right.$$

Thus the shock density and network are symmetric with respect to nodes $1, 2, 3$. We can show the following under these circumstances:

Proposition 4 *Make Assumption 2, then the symmetric market-integration function is a constant given by:*

$$\mu = \frac{3(n\bar{q})^2 + 12Kn\bar{q} + 12K^2}{3(n\bar{q})^2 + 8Kn\bar{q} + 4K^2}. \tag{41}$$

Solutions to (25) are SFE, and the unique inverse supply function of each firm in nodes $i \in \{1, 2, 3\}$ is given by (32).

Proof. See Appendix B. ■

Figure 3 and Corollary 2 apply to the star network as well. It is only the market integration function that depends on whether the network has two nodes or is star shaped.

4 Meshed network

So far we have studied radial networks, where there is a unique path between every pair of nodes. Now we generalize our results to include more complicated networks consisting of M nodes and N arcs, where $N \geq M$. This means that there will be at least one cycle in the network and there will be at least two paths between any two nodes in the cycle [11]. Thus we need to make assumptions of how the transport route is chosen for cases when there are multiple possible paths. Similar to Wilson [45] we assume that flows are determined by physical laws that are valid for electricity and incompressible mediums with laminar (non-turbulent) flows. Such flows are sometimes called potential flows, because one can model them as being driven by the potentials $\boldsymbol{\phi}$ in the nodes. In case the commodity is a gas or liquid (e.g. oil), the potential is the pressure at the node. In a DC network it is the voltage that is the potential.[22] For DC networks and laminar flows it can be shown that the electricity and flow choose paths that minimizes total losses.

In a potential flow model, the flow in the arc k is the result of the potential difference between its endpoints. Given a vector of potentials $\boldsymbol{\phi}$, we have

$$t_k = \frac{-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k}{X_k} \tag{42}$$

where $-\left(\mathbf{A}^\top \boldsymbol{\phi}\right)_k$ is the potential difference and X_k is the impedance resisting the the flow through the arc. The impedance parameter is determined by the geometrical and material properties of the line/pipe that transports the commodity, and is independent of the flow in the arc. In a DC network, the impedance is given by the resistance of the line.[23]

The matrix $\mathbf{A}$ has rank $M-1$, so the potentials $\boldsymbol{\phi}$ are not uniquely defined by (42). Thus we can arbitrarily choose one node (say i) and set its potential ϕ_i arbitrarily. Similar to Wilson [45], we set the potential of this *swing node* to zero. This corresponds to deleting row i from $\mathbf{A}$ to form the matrix $\mathbf{A}_{-i}$ with rank $M-1$ [41]. To simplify the analysis we rule out some unrealistic or unlikely cases: we assume that the impedance is positive and that the capacities of the arcs and impedance factors are such that for any feasible flow, the set of arcs with flows at a lower or upper bound contains no cycles. [24]

As in the radial case, the market is cleared when all profitable feasible arbitrage trades have been exhausted. Similar to Wilson [45] we solve for this outcome by

[22] For AC networks it is standard to calculate electric power flows by means of a *DC-load flow approximation*, where $\boldsymbol{\phi}$ is the vector of voltage phase angles at the nodes [15].

[23] In a DC-load flow approximation of an AC network, X_k represents the reactance of the transmission line.

[24] This precludes certain degenerate solutions which can only arise if the values of the bounds and impedances for arcs forming a loop $\mathcal{L}$, satisfy equations of the form

$$\sum_{k \in \mathcal{L}} \delta_k X_k K_k = 0$$

where $\delta_k = 1$ if arc k is oriented in the direction that $\mathcal{L}$ is traversed and $\delta_k = -1$ otherwise. We can preclude instances having such solutions by perturbing the line capacities if necessary.

calculating feasible production, consumption and transportation that maximizes stated social welfare, i.e. the social welfare that would occur if producers' supply functions corresponded to their true costs. In the literature this is referred to as an economic dispatch problem (EDP) [15][17].

$$\begin{aligned}
\text{EDP:}\quad &\text{minimize} && \textstyle\sum_{i=1}^{M}\sum_{g=1}^{n_i}\int_0^{q_{ig}} Q_{ig}^{-1}(x)dx - \sum_{i=1}^{M}\int_0^{y_i} D_i^{-1}(y)dy \\
&\text{subject to} && \mathbf{At}+\mathbf{q}-\mathbf{y}=\boldsymbol{\varepsilon}, && [\mathbf{p}] \\
& && -\mathbf{K}\le\mathbf{t}\le\mathbf{K}, && [\boldsymbol{\sigma},\boldsymbol{\rho}] \\
& && \mathbf{Xt}=-\mathbf{A}^\top\boldsymbol{\phi}. && [\boldsymbol{\lambda}],
\end{aligned}$$

The shadow prices for the constraints are shown on the right-hand side in brackets. The Karush-Kuhn-Tucker conditions of EDP are

$$\begin{aligned}
\text{KKT:}\quad & \mathbf{A}^\top\mathbf{p}+\mathbf{X}^\top\boldsymbol{\lambda}=\boldsymbol{\rho}-\boldsymbol{\sigma} \\
& 0\le\boldsymbol{\rho}\perp\mathbf{K}-\mathbf{t}\ge\mathbf{0} \\
& 0\le\boldsymbol{\sigma}\perp\mathbf{K}+\mathbf{t}\ge\mathbf{0} \\
& \mathbf{A}\boldsymbol{\lambda}=\mathbf{0} \\
& \mathbf{At}+\mathbf{s}(\mathbf{p})=\boldsymbol{\varepsilon} \\
& \mathbf{Xt}=-\mathbf{A}^\top\boldsymbol{\phi}
\end{aligned}$$

In radial networks the columns of the matrix $\mathbf{A}$ correspond to network arcs defining a tree, and so they are linearly independent (see [41]). This means that $\mathbf{A}\boldsymbol{\lambda}=\mathbf{0}$ has a unique solution $\boldsymbol{\lambda}=\mathbf{0}$, which allows $\boldsymbol{\lambda}$ to be removed from the market clearing conditions. In this case the conditions become the same as those for radial networks in (7).

We now return to discuss the general case. The prices $\mathbf{p}$ that satisfy the KKT conditions in any congestion state ω must meet certain conditions. First observe that since $\mathbf{X}$ is diagonal and nonsingular, the following can be obtained from the first KKT condition:

$$\mathbf{X}^{-1}\mathbf{A}^\top\mathbf{p}+\boldsymbol{\lambda}=\mathbf{X}^{-1}(\boldsymbol{\rho}-\boldsymbol{\sigma}) \tag{43}$$

Multiplying by $\mathbf{A}$ and using the KKT condition that $\mathbf{A}\boldsymbol{\lambda}=\mathbf{0}$ yields

$$\mathbf{AX}^{-1}\mathbf{A}^\top\mathbf{p}=\mathbf{AX}^{-1}(\boldsymbol{\rho}-\boldsymbol{\sigma}) \tag{44}$$

In the context of power system networks the matrix $\mathbf{AX}^{-1}\mathbf{A}^\top$ is called a network admittance matrix, and when $\mathbf{X}$ is the identity it is a *Laplacian matrix*. The matrix $\mathbf{AX}^{-1}\mathbf{A}^\top$ has rank $M-1$, so the vector of prices $\mathbf{p}$ is not uniquely determined by the choice of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$. Recall that $\mathbf{A}_i$ is row i of matrix $\mathbf{A}$, and $\mathbf{A}_{-i}$ is matrix $\mathbf{A}$ with row i eliminated. As in section 3 we choose a node i, say, as trading hub and assign its price to be p.

Lemma 4 *Nodal prices $\mathbf{p}_{-i}$ can be expressed in terms of the price, p, of the trading hub and the shadow prices.*

$$\mathbf{p}_{-i}=p\mathbf{1}_{M-1}+\mathbf{E}(\boldsymbol{\rho}-\boldsymbol{\sigma}), \tag{45}$$

where

$$\mathbf{E} = (\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T})^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}. \tag{46}$$

Proof. We can write (44) as follows:

$$\mathbf{A}\mathbf{X}^{-1}\left(\left(\mathbf{A}_{-i}\right)^{T}\mathbf{p}_{-i} + p\left(\mathbf{A}_{i}\right)^{T}\right) = \mathbf{A}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}).$$

We can remove row i from this equation, multiply by $(\mathbf{A}_{-i}\mathbf{X}^{-1}\mathbf{A}_{-i}^{\top})^{-1}$ and make rearrangements, so that

$$\begin{aligned}
\mathbf{p}_{-i} &= -(\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T})^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{i}\right)^{T}p \\
&\quad +(\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T})^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}) \\
&= p\mathbf{1}_{M-1} + (\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T})^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}(\boldsymbol{\rho} - \boldsymbol{\sigma}).
\end{aligned} \tag{47}$$

The last step follows as the columns of $\mathbf{A}^{T}$ sum to a column vector of zeros, so

$$\begin{gathered}
\left(\mathbf{A}_{-i}\right)^{T}\mathbf{1}_{M-1} + \left(\mathbf{A}_{i}\right)^{T} = \mathbf{0}_{M-1} \\
\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T}\mathbf{1}_{M-1} + \mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{i}\right)^{T} = \mathbf{0}_{M-1} \\
\mathbf{1}_{M-1} = -(\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{-i}\right)^{T})^{-1}\mathbf{A}_{-i}\mathbf{X}^{-1}\left(\mathbf{A}_{i}\right)^{T}.
\end{gathered} \tag{48}$$

(45) and (46) follows from (47). ■

Recall that $\mathbf{A}_{-i}$ is nonsingular in the radial case, which gives $\mathbf{E} = (\left(\mathbf{A}_{-i}\right)^{T})^{-1}$ as in (8). More generally, $\mathbf{A}_{-i}$ will have $M-1$ rows and $N > M-1$ columns, and so it will not have an inverse. As in the radial case, we denote by $\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma})$ the vector of nodal prices defined by (45), where we choose to suppress the dependence on ω for notational convenience. We let H be a matrix with $N-(M-1)$ rows forming a basis for the null space of A. H could for example be the rows of the orientation vectors of a set of $N-(M-1)$ cycles in the network [41]. As for the radial case we have $S(\omega) = L(\omega) \times U(\omega) \times B(\omega)$. However, $B(\omega)$ is more complicated in the meshed case:

$$B(\omega) = \{\mathbf{t}_{B} : \mathbf{Y}_{B}\mathbf{t}_{B} = -\mathbf{Y}_{L}\mathbf{t}_{L} - \mathbf{Y}_{U}\mathbf{t}_{U}, \quad -\mathbf{K}_{B} \leq \mathbf{t}_{B} \leq \mathbf{K}_{B}\}, \tag{49}$$

where $Y = HX$.

Lemma 5

$$\frac{\partial\psi_{i,g}(p,q)}{\partial p} = \sum_{\omega}\int_{S(\omega)} f\left(\mathbf{A}\mathbf{t} + \mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),q\right)\right) J_{p}(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}, \tag{50}$$

$$\frac{\partial\psi_{i,g}(p,q)}{\partial q} = \sum_{\omega}\int_{S(\omega)} f\left(\mathbf{A}\mathbf{t} + \mathbf{s}\left(\mathbf{p}(p,\boldsymbol{\rho},\boldsymbol{\sigma}),q\right)\right) J_{q}(\omega)\mathbf{dt}_{B(\omega)}\mathbf{d}\boldsymbol{\rho}_{U(\omega)}\mathbf{d}\boldsymbol{\sigma}_{L(\omega)}, \tag{51}$$

Proof. By definition we have $\mathbf{A}\mathbf{H}^\top = \mathbf{0}$, it follows for any $\boldsymbol{\phi}$ that

$$\mathbf{H}\mathbf{A}^\top\boldsymbol{\phi} = \mathbf{0}.$$

Now the KKT conditions amount to:

$$\begin{aligned}\boldsymbol{\varepsilon} &= \mathbf{A}\mathbf{t} + \mathbf{s}(p\mathbf{1}_{M-1} + \mathbf{E}(\boldsymbol{\rho} - \boldsymbol{\sigma})) \\ \mathbf{t} &\in [-K, K] \\ \mathbf{H}\mathbf{X}\mathbf{t} &= -\mathbf{H}\mathbf{A}^\top\boldsymbol{\phi} = 0.\end{aligned}$$

We seek M degrees of freedom in these equations that will specify a range over which to integrate $\boldsymbol{\varepsilon}$. One free variable is given by either the price in node i, p, or the supply of firm g, q. The remaining $M-1$ degrees of freedom are integrated in the $\mathbf{t}$, $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ space. If every $\mathbf{t} \in (-K, K)$ then $\boldsymbol{\rho} = \boldsymbol{\sigma} = \mathbf{0}$, and we have N variables and $N - (M-1)$ constraints from $\mathbf{H}\mathbf{X}\mathbf{t} = 0$, so we are left with $M-1$ variables to integrate with. For every component of $\mathbf{t}$ that is at a bound, we get a non-negative component of $\boldsymbol{\rho}$ or a component of $\boldsymbol{\sigma}$ that is free to leave its bound. We partition $\mathbf{Y} = \mathbf{H}\mathbf{X}$ into $\mathbf{Y}_L$, $\mathbf{Y}_B$ and $\mathbf{Y}_U$ corresponding to flows at the lower bound, between bounds and at the upper bound. We have $\mathbf{Y}\mathbf{t} = \mathbf{0}$, so to integrate over a congestion state ω we fix constrained components ($\mathbf{t}_L = -\mathbf{K}$ and $\mathbf{t}_U = \mathbf{K}$) of $\mathbf{t}$ to get

$$\mathbf{Y}_B\mathbf{t}_B = -\mathbf{Y}_L\mathbf{t}_L - \mathbf{Y}_U\mathbf{t}_U$$

and free unconstrained components of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ to get $\boldsymbol{\sigma}_L$ and $\boldsymbol{\rho}_U$. Similar to the proof of Proposition 1, we get (50) and (51) by first keeping q fixed and then p. ■

$J_p(\omega)$ and $J_q(\omega)$ are defined by (18) and (21), respectively. The expressions (50) and (51) can be substituted into (5) and (6) to give optimality conditions in a meshed network. Unfortunately, the determinants $J_p(\omega)$ and $J_q(\omega)$ do not simplify as in the radial case. In the radial case, each agent effectively faces a probability-weighted residual demand curve defined by Corollary 1. In the meshed case the residual demand curve in a congestion state ω involves combinations of the slopes of competitors' supply functions measured at different prices. In other words, nodes in a meshed market may be integrated in a congestion state in the sense that transport between their nodes is uncongested (with some adjustment in dispatch) but still experience different prices. This makes an analytical derivation of equilibrium a lot more challenging in mesh networks, except for some special cases (such as when all strategic agents are located in the same uncongested region). Numerical solutions to the optimality conditions could potentially be obtained for more general cases.

5 Alternative market designs and strategies

Finally, we want to briefly note that our expressions in Sections 3.1 and 4 for how market distribution functions can be calculated in radial and meshed networks are not restricted to SFE in networks with nodal pricing. They can also be used to calculate Cournot NE in networks with additive demand shocks. We know from

Anderson and Philpott [4] that the optimality condition of a vertical offer q from firm g in node i facing an uncertain residual demand is:

$$\int_0^{\overline{p}} Z(p,q)\,dp = 0$$

with the second-order condition that $\int_0^{\overline{p}} Z_q(p,q)\,dp \leq 0$. For radial networks with Cournot competition, $Z(p,q)$ can be calculated as in Proposition 1 if one sets the slope of net-supply in each node equal to the nodal demand slope.

Our approach is not limited to cases with local marginal prices. As long as arbitrageurs in the transport sector or the regulated network operator are price-takers, it is often straightforward to adjust our optimality conditions to networks with other auction formats. For example, consider networks with discriminatory (pay-as-bid) pricing as in the electricity market of Britain. Anderson et al. [3] show that the optimality condition of a firm's offer in such an auction is given by [25]

$$Z = \frac{\partial \psi_{ig}}{\partial p}(p - C'_{ig}(q)) - 1 + \psi_{ig}(p,q),$$

and the same conditions as in (6). For a radial network, $\frac{\partial \psi_{ig}}{\partial p}$ and $\psi_{ig}(p,q)$ are given by (19) and (20), respectively. $J_p(\omega)$ can be calculated as in Lemma 7 (see Appendix A).

6 Conclusions

We derive optimality conditions for firms offering supply functions into a network with transport constraints and local additive demand shocks. In principle, a system of such optimality conditions can be used to numerically calculate asymmetric Supply Function Equilibria (SFE) in a general network. In the paper, we focus on characterizing symmetric SFE in symmetric radial networks. We verify that monotonic solutions to the first-order conditions are Supply Function Equilibria (SFE) when the joint probability density of the local demand shocks is sufficiently evenly distributed, i.e. close to a uniform multi-dimensional distribution. But existence of SFE cannot be taken for granted. Perfectly correlated shocks or steep slopes and discontinuities in the shock density will not smooth the kinks in the residual demand curves sufficiently well, and then profitable deviations from the first-order solution will exist.

In an isolated node with one-dimensional additive demand shocks, the optimal output of a producer is proportional to its mark-up and the slope of the residual demand that it is facing. We show that in a network with multi-dimensional shocks, this generalizes: the optimal output of a producer is proportional to its mark-up and the expected slope of the residual demand that it is facing. Thus the probability with which the producer's node is completely integrated with other

[25] Note that we have changed the sign of the Z function in Anderson et al [3] for pay-as-bid markets to keep it consistent with the Z function used in this paper.

nodes, i.e. connected to other nodes via uncongested arcs, is of great importance for the optimal offer.

For symmetric equilibria it is useful to define a market integration function, which equals the expected number of nodes that are completely integrated with a particular node; a node is always completely integrated with itself. Firms' supply functions depend on the number of firms in the market. Still it can be shown that in a symmetric equilibrium with inelastic demand, market integration is a function of the total nodal production in a node. The function can be determined from exogenous parameters: the network topology, the demand shock distribution and production and transport capacities. The implication is that oligopoly producers will have high mark-ups at output levels for which the market integration function returns small values, and lower mark-ups at output levels where market integration is expected to be high.

The market integration function simplifies to a constant for inelastic demand with multi-dimensional uniformly distributed shocks in symmetric radial networks. We use our optimality conditions to explicitly solve for symmetric equilibria in two-node and star networks for such shocks. We show that an equilibrium offer in a node of such a network is identical to the equilibrium offer in an isolated node where the number of symmetric firms per node have been scaled by the market integration function. Thus previous results for symmetric SFE in single node networks become applicable to symmetric SFE in symmetric radial networks with transport constraints. We also show that these symmetric equilibria are well-behaved: (i) mark-ups are positive for a positive output, and (ii) for a given total production cost, mark-ups decrease with more firms in the market.

We focus on characterising SFE in radial networks, but we also show how our optimality conditions can be generalized to consider meshed networks, albeit with a significant increase in complexity. We also present optimality conditions for SFE in networks with discriminatory pricing and Cournot NE in networks with uncertain demand.

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Appendix A: Properties of node-arc incidence matrices and their implications for radial networks

We start the appendix by exploring some special properties of the node-arc incidence matrix $\mathbf{A}$ for a radial network. We have the following technical results.

Lemma 6 *Suppose* $\mathbf{A}$ *is the node-arc incidence matrix for a radial network with* M *nodes. If* $j < M$ *then* $\det \mathbf{A}_{-j} = -\det \mathbf{A}_{-(j+1)}$

Proof. Introduce a new matrix $\mathbf{W}$ which is identical to $\mathbf{A}_{-j}$, except that row j of $\mathbf{A}_{-j}$, which is equal to $\mathbf{A}_{j+1}$, has been replaced by the sum of all rows in $\mathbf{A}_{-j}$. Such manipulations are allowed without changing the determinant [41], so $\det(\mathbf{W}) = \det \mathbf{A}_{-j}$. Node-arc incidence matrices are such that $\mathbf{W}_j$, the sum of all rows in $\mathbf{A}_{-j}$, is equal to $-\mathbf{A}_j$ (row j of $\mathbf{A}$). Thus $\mathbf{W}$ is identical to $\mathbf{A}_{-(j+1)}$ except that elements have opposite signs in row j. In the calculation of the determinants we can expand them along row j of both $\mathbf{W}$ and $\mathbf{A}_{-(j+1)}$, which gives the stated result [41]. ■

By applying Lemma 6 $|k-j|$ times we get the following result.

Corollary 3 *If* $\mathbf{A}$ *is the node-arc incidence matrix for a radial network then* $(-1)^j \det \mathbf{A}_{-j} = (-1)^k \det \mathbf{A}_{-k}$

In the three lemmas below we use Corollary 3 and other properties of node-arc incidence matrices to derive explicit expressions for the substitution factors $J_p(\omega)$ and $J_q(\omega)$ in (18) and (21), respectively.

Lemma 7 $J_p(\omega) = \left(S'_{i,-g}(\pi) + \sum_{k\in\Xi(\omega)\setminus i} S'_k(\pi)\right) J_F(\omega)$, *where*

$$J_F(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)})} \right|. \tag{52}$$

Proof. We use π to denote the nodal price of the trading hub. Thus, given a state ω, the price in node i and all nodes $j \in \Xi(\omega)$ is π (irrespective of $\boldsymbol{\rho}_{U(\omega)}$ and $\boldsymbol{\sigma}_{L(\omega)}$). Thus it follows from (2), (13) and (9) that

$$\frac{\partial \varepsilon_j}{\partial \rho_k} = s'_j(p_j)\frac{\partial p_j}{\partial \rho_k} = 0, \text{ if } j \in \Xi(\omega) \text{ and } k \in U(\omega) \tag{53}$$

$$\frac{\partial \varepsilon_j}{\partial \sigma_k} = s'_j(p_j)\frac{\partial p_j}{\partial \sigma_k} = 0, \text{ if } j \in \Xi(\omega) \text{ and } k \in L(\omega) \tag{54}$$

$$\frac{\partial \varepsilon_j}{\partial \pi} = \begin{cases} S'_j(\pi) & \text{if } j \in \Xi(\omega)\setminus i \\ S'_{j,-g}(\pi) & \text{if } j = i. \end{cases} \tag{55}$$

The nodal flow balance in (2) can be written as follows:

$$\begin{aligned} \mathbf{A}_{\Xi(\omega)}\mathbf{t}_{\Xi(\omega)} + \mathbf{s}_{\Xi(\omega)}(\mathbf{p}) &= \boldsymbol{\varepsilon}_{\Xi(\omega)} \\ \mathbf{A}_{F(\omega)}\mathbf{t}_{F(\omega)} + \mathbf{s}_{F(\omega)}(\mathbf{p}) &= \boldsymbol{\varepsilon}_{F(\omega)}. \end{aligned}$$

Thus

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{\Xi(\omega)}\right)_k}{\partial\left(\mathbf{t}_{\Xi(\omega)}\right)_j} = \left(\mathbf{A}_{\Xi(\omega)}\right)_{kj} \tag{56}$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{\Xi(\omega)}\right)_k}{\partial\left(\mathbf{t}^F_{B(\omega)}\right)_j} = 0 \tag{57}$$

and

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{F(\omega)}\right)_k}{\partial\left(\mathbf{t}_{F(\omega)}\right)_j} = \left(\mathbf{A}_{F(\omega)}\right)_{kj} \tag{58}$$

$$\frac{\partial\left(\boldsymbol{\varepsilon}_{F(\omega)}\right)_k}{\partial\left(\mathbf{t}_{\Xi(\omega)}\right)_j} = 0. \tag{59}$$

From (53)-(59) we realize that

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \pi\right)} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial \pi} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \pi} \end{bmatrix} \tag{60}$$

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^F_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} \end{bmatrix}. \tag{61}$$

When calculating $J_p(\omega) = \left|\frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \pi\right)}\right|$, we will expand the determinant along the Mth column in (60) with entries $\frac{\partial \varepsilon_k}{\partial \pi}$ giving the net-supply slopes as shown in (55). It follows from (55) and the definition of the determinant that [41]:

$$J_p(\omega) = \left|\sum_k (-1)^{k+M} \frac{\partial \varepsilon_k}{\partial \pi} \det\left(\mathbf{B}_{-k}\right)\right|.$$

$\mathbf{A}_{\Xi(\omega)}$ is the node arc incidence matrix of a connected radial network. This matrix has linearly dependent rows and has rank $M_{\Xi(\omega)} - 1$. Thus it follows from (61) that $\det\left(\mathbf{B}_{-k}\right) = 0$ if $k \in F(\omega)$. If $k \in \Xi(\omega)$ then $\mathbf{B}_{-k}$ is a block matrix with determinant $\left|\left(\mathbf{A}_{\Xi(\omega)}\right)_{-k}\right| J_F(\omega)$. Thus $J_p(\omega)$ can be written as

$$\begin{aligned} J_F(\omega) & \left|(-1)^{i+M} S'_{i,-g}(\pi) \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i} + \textstyle\sum_{k \in \Xi(\omega) \backslash i} (-1)^{k+M} S'_k(\pi) \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-k}\right| \\ & = J_F(\omega)\left(S'_{i,-g}(\pi) + \textstyle\sum_{k \in \Xi(\omega) \backslash i} S'_k(\pi)\right)\left|(-1)^{M_{\Xi(\omega)}} \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i}\right| \end{aligned}$$

by Corollary 3 and the monotonicity of net-supply functions. Now, since $\mathbf{A}_{\Xi(\omega)}$ is the node-arc incidence matrix of a connected radial network, it follows from Bapat [10] (p. 13) that $\left|(-1)^{M_{\Xi(\omega)}} \det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-j}\right|$ is 1. ■

Lemma 8 $J_q(\omega) = J_F(\omega)$.

Proof. We use r to denote the output of firm g in node i. Thus, we have from (2) and (13) that

$$\frac{\partial \varepsilon_k}{\partial r} = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i. \end{cases}$$

Similar to (60) we have

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, r\right)} = \begin{bmatrix} \mathbf{A}_{\Xi(\omega)} & \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi(\omega)}}{\partial r} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}^{F}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} & \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial r} \end{bmatrix}. \qquad (62)$$

As in the proof of Lemma 7, we expand the determinant $\left|\frac{\partial \boldsymbol{\varepsilon}}{\partial\left(\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, r\right)}\right|$ along the Mth column, which has zeros in rows $k \neq i$ and a one in row i. We use the definition of $\mathbf{B}$ in (61), so it follows from the definition of the determinant that [41]:

$$\begin{aligned} J_q(\omega) &= \left|(-1)^{i+M} \det\left(\mathbf{B}_{-i}\right)\right| = \left|\det\left(\mathbf{B}_{-i}\right)\right| \\ &= \left|\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i}\right| J_F(\omega) \end{aligned}$$

because $\mathbf{B}_{-i}$ is a block matrix with determinant $\left|\left(\mathbf{A}_{\Xi(\omega)}\right)_{-k}\right| J_F(\omega)$. $\mathbf{A}_{\Xi(\omega)}$ is a node-arc incidence matrix of a connected radial network. Thus it follows from Bapat [10] (p. 13) that $\left|\det\left(\mathbf{A}_{\Xi(\omega)}\right)_{-i}\right|$ is 1, which gives the stated result. ■

Lemma 9 *Row k of the Jacobian matrix $\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\mathbf{t}^{F}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}$ can be constructed as follows for the state ω:*

$$\left(\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\mathbf{t}^{F}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)}\right)_k = \left[\; \left(\mathbf{A}^{F}_{B(\omega)}\right)_k \quad S'_k\left(p_k\right)\left(\mathbf{E}_{U(\omega)}\right)_k \quad -S'_k\left(p_k\right)\left(\mathbf{E}_{L(\omega)}\right)_k \;\right] \qquad (63)$$

for $k \in F(\omega)$.

Proof. We partition the columns of $\mathbf{A}_{F(\omega)}$ into $\mathbf{A}^{F}_{L(\omega)}$, $\mathbf{A}^{F}_{B(\omega)}$ and $\mathbf{A}^{F}_{U(\omega)}$, corresponding to flows $\mathbf{t}_F$ being at their lower bounds, strictly between their bounds, and at their upper bounds. Thus the flow balance in (2) can be written as follows

$$\mathbf{A}^{F}_{B(\omega)}\mathbf{t}^{F}_{B(\omega)} + \mathbf{A}^{F}_{U(\omega)}\mathbf{t}_{U(\omega)} + \mathbf{A}^{F}_{L(\omega)}\mathbf{t}_{L(\omega)} + \mathbf{s}_{F(\omega)}\left(\mathbf{p}\right) = \boldsymbol{\varepsilon}_{F(\omega)}. \qquad (64)$$

Observe that (9) implies that

$$\frac{\partial \varepsilon^{F}_k}{\partial \rho_j} = \frac{\partial \varepsilon^{F}_k}{\partial p_k}\frac{\partial p_k}{\partial \rho_j} = S'_k\left(p_k\right)\left(\mathbf{E}_{U(\omega)}\right)_{kj}$$

and

$$\frac{\partial \varepsilon^{F}_k}{\partial \sigma_j} = \frac{\partial \varepsilon^{F}_k}{\partial p_k}\frac{\partial p_k}{\partial \sigma_j} = -S'_k\left(p_k\right)\left(\mathbf{E}_{L(\omega)}\right)_{kj}$$

which gives the result. ■

Appendix B: Selected proofs

Proof. (Lemma 3). Below we list the congestion states of the network and how we partition the nodes for each state: ■

State	t	ρ	σ	Ξ	F
ω_1	$\in (-K, K)$	0	0	$\{1,2\}$	$\emptyset$
ω_2	K	$\in [0, \infty)$	0	$\{1\}$	$\{2\}$
ω_3	$-K$	0	$\in [0, \infty)$	$\{1\}$	$\{2\}$

We have from (2) that

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{A}} t. \tag{65}$$

We have from (65) that

State	$\mathbf{A}_\Xi$	$\mathbf{A}_F$	$\mathbf{A}_B^F$
ω_1	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\emptyset$	$\emptyset$
ω_2	0	1	$\emptyset$
ω_3	0	1	$\emptyset$

We also have

$$\mathbf{A}_{-1} = 1 = (\mathbf{A}_{-1})^T = \left((\mathbf{A}_{-1})^T\right)^{-1} = \mathbf{E}.$$

Thus

State	$\mathbf{E}_\Xi$	$\mathbf{E}_F$	$\mathbf{E}_U^F$	$\mathbf{E}_L^F$
ω_1	1	$\emptyset$	$\emptyset$	$\emptyset$
ω_2	$\emptyset$	1	1	$\emptyset$
ω_3	$\emptyset$	1	$\emptyset$	1

We set $p_1 = \pi$, so it follows from (10) that

$$p_2 = \pi + \rho - \sigma. \tag{66}$$

The network is completely integrated in state ω_1, so $\boldsymbol{\varepsilon}_{F(\omega_1)}$ is empty. We only need the substitution factor $J_F(\omega)$ for states ω_2 and ω_3. It follows from (17) and (66) that

$$J_F(\omega_2) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega_2)}}{\partial \left(\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega}_2)}\right)} \right| = S_2'(p_2) = S_2'(\pi + \rho)$$

$$J_F(\omega_3) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega_3)}}{\partial \left(\boldsymbol{\sigma}_{\mathbf{L}(\omega_3)}\right)} \right| = |-S_2'(p_2)| = S_2'(\pi - \sigma).$$

(15) now yields:

$$P(p, q, \omega_1) = \int_{-K}^{K} f(\mathbf{A}t + \mathbf{s}(\mathbf{p}, q))\, dt = \int_{-K}^{K} f(q + S_{1,-g}(p) - t, S_2(p) + t)\, dt,$$

$$P(p,q,\omega_2) = \int_0^\infty f(\mathbf{A}t+\mathbf{s}(\mathbf{p},q))\, J_F(\omega_2) d\rho$$
$$= \int_0^\infty f(q + S_{1,-g}(p) - K, S_2(p+\rho) + K)\, S_2'(p+\rho)\, d\rho$$

and

$$P(p,q,\omega_3) = \int_0^\infty f(\mathbf{A}t+\mathbf{s}(\mathbf{p},q))\, J_F(\omega_3) d\sigma$$
$$= \int_0^\infty f(q + S_{1,-g}(p) + K, S_2(p-\sigma) - K)\, S_2'(p-\sigma)\, d\sigma.$$

This gives us (27) after the substitutions $\varepsilon_2 = S_2(p+\rho)+K$ and $\varepsilon_2 = S_2(p-\sigma)-K$, respectively, have been applied to the integrals of the states ω_2 and ω_3. The equation (26) follows from (14) and that the two nodes are only completely integrated in state ω_1.

Proof. (Proposition 2). Symmetry of the network, costs and shock densities ensure that the optimal supply functions of all producers are given by identical optimality conditions. We have $S_2(p) = q + S_{1,-g}(p) = nQ(p)$ in a symmetric equilibrium with inelastic demand, so (30) follows from (27). The differential equation in the statement follows from (26). In case that production capacity would bind at some price $p_b < \overline{p}$ then $Q(p)$ is inelastic in the range $(p_b, \overline{p})$, and it follows from (26) that $Z(p,q) < 0$ when $q < \overline{q}$ and $p \in (p_b, \overline{p})$. This would violate the second-order condition in (6), and it is necessary that this condition is locally satisfied [4]. Thus the production capacity must bind at the reservation price, which gives our initial condition.

Next we show that the solution is unique. It follows from the assumptions for $f(\varepsilon_1, \varepsilon_2)$, our definition of $P(nQ,\omega)$ and from Definition 1 that

$$\frac{1}{(n\mu(nQ)-1)} > 0,$$

and that $\frac{1}{(n\mu(nQ)-1)}$ is Lipschitz continuous in Q. Consider a price $\tilde{p} \in (C'(0), \overline{p})$. We now want to show that $p - C'(Q(p))$ is bounded away from zero in the range $[\tilde{p}, \overline{p}]$. This is obvious for constant marginal costs, as we then have that $\tilde{p} - C'(Q(\tilde{p})) = \tilde{p} - C'(0) > 0$. For strictly increasing marginal costs we can use the following argument. It follows from Picard-Lindelöf's theorem and $\overline{p} > C'(\overline{q})$ that a unique solution to (28) must exist for some range $[p_0, \overline{p}]$. In this price range the mark-up, $p - C'(Q(p))$, is smallest at some price p^* where the inverse supply function is at least as steep as the marginal cost curve, i.e. $Q'(p^*) \leq \frac{1}{C''(Q(p^*))}$. Thus it follows from (28) that

$$p^* - C'(Q(p^*)) \geq \frac{Q(p^*)C''(Q(p^*))}{(n\mu(nQ(p^*))-1)}.$$

This is bounded away from zero whenever $Q(p^*)$ is bounded away from zero if marginal costs are strictly increasing. In case $Q(p^*) = 0$ for some price $p^* > C'(0)$, it follows from (28) that $Q'(p) = 0$ for $p \in (\tilde{p}, p^*)$. Thus it follows from Picard-Lindelöf's theorem and the properties of (28) that a unique monotonic symmetric solution will exist for the price interval $[\tilde{p}, \overline{p}]$. We can repeat the argument for any $\tilde{p} \in (C'(0), \overline{p})$ to show that a unique monotonic symmetric solution will exist for the price interval $(C'(0), \overline{p}]$.

We now verify the global second order conditions. To simplify notation let

$$\alpha(p,q) = P(p,q,\omega_1) + P(p,q,\omega_2) + P(p,q,\omega_3), \tag{67}$$
$$\beta(p,q) = (2n-1)P(p,q,\omega_1) + (n-1)P(p,q,\omega_2) + (n-1)P(p,q,\omega_3) \tag{68}$$

We have from (26) that

$$Z(p,q) = (p - C'(q))\beta(p,q)Q'(p) - q\alpha(p,q)$$

We also have $C'' \geq 0$ and $Q'(p) \geq 0$, so

$$Z_q \leq (p - C'(q))\beta_q Q' - \alpha - q\alpha_q.$$

Further, whenever $Z(p,q) = 0$, we have

$$Z_q \leq \frac{q\alpha\beta_q - \beta\alpha - q\beta\alpha_q}{\beta}.$$

We know from (6) that the solution is an equilibrium if $Z(p,q) \geq 0$ when $q \leq Q(p)$ and $Z(p,q) \leq 0$ when $q \geq Q(p)$. This follows if $Z_q(p,q) \leq 0$ whenever $Z(p,q) = 0$. To verify this sufficiency condition, it suffices to show that

$$\beta(p,q)\alpha(p,q) + q\beta(p,q)\alpha_q(p,q) - q\alpha(p,q)\beta_q(p,q) \geq 0. \tag{69}$$

To show this observe that the assumption

$$2n\overline{q}\left|f_i(\varepsilon_1,\varepsilon_2)\right| \leq (3n-2)\, f(\varepsilon_1,\varepsilon_2)$$

implies from (27) that

$$\begin{aligned}
2nq\left|P_q(p,q,\omega_1)\right| &= 2nq\left|\int_{-K}^{K}\frac{\partial}{\partial q}f\left(q+S_{1,-g}(p)-t, S_2(p)+t\right)dt\right| \\
&\leq 2n\overline{q}\left|\int_{-K}^{K}\frac{\partial}{\partial q}f\left(q+S_{1,-g}(p)-t, S_2(p)+t\right)dt\right| \\
&\leq \int_{-K}^{K}2n\overline{q}\left|f_1\left(q+S_{1,-g}(p)-t, S_2(p)+t\right)\right|dt \\
&\leq (3n-2)\int_{-K}^{K}f\left(q+S_{1,-g}(p)-t, S_2(p)+t\right)dt \\
&= (3n-2)\,P(p,q,\omega_1).
\end{aligned}$$

Similarly $2nq\left|P_q(p,q,\omega_3)\right| \leq (3n-2)\,P(p,q,\omega_3)$ and $2nq\left|P_q(p,q,\omega_2)\right| \leq (3n-2)\,P(p,q,\omega_2)$. It follows from (67) and (68) that

$$\begin{aligned}
& q\beta(p,q)\alpha_q(p,q) - q\alpha(p,q)\beta_q(p,q) \\
=\ & qn\left(P(p,q,\omega_1)\left(P_q(p,q,\omega_2)+P_q(p,q,\omega_3)\right)\right) - qnP_q(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right)) \\
\geq\ & -(3n-2)P(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right).
\end{aligned}$$

It can be deduced from (67) and (68) that

$$\beta(p,q)\alpha(p,q) \geq (3n-2)P(p,q,\omega_1)\left(P(p,q,\omega_2)+P(p,q,\omega_3)\right).$$

Thus (69) is satisfied, which is sufficient for an equilibrium. ■

Proof. (Proposition 3) It follows from the definitions of $P(p,q,\omega_1)$, $P(p,q,\omega_2)$ and $P(p,q,\omega_3)$ in (27) that under Assumption 1 we get:

$$\begin{aligned}
P(p,q,\omega_1) &= \int_{-K}^{K} f\left(q+S_{1,-g}(p)-t, S_2(p)+t\right) dt = \int_{-K}^{K} \tfrac{dt}{V_1} = \tfrac{2K}{V_1} \\
P(p,q,\omega_2) &= \int_{S_2(p)+K}^{\infty} f\left(q+S_{1,-g}(p)-K, \varepsilon_2\right) d\varepsilon_2 = \int_{S_2(p)+K}^{n\bar{q}+K} \tfrac{d\varepsilon_2}{V_1} = \tfrac{n\bar{q}-S_2(p)}{V_1} \\
P(p,q,\omega_3) &= \int_{-\infty}^{S_2(p)-K} f\left(q+S_{1,-g}(p)+K, \varepsilon_2\right) d\varepsilon_2 = \int_{-K}^{S_2(p)-K} \tfrac{d\varepsilon_2}{V_1} = \tfrac{S_2(p)}{V_1}.
\end{aligned} \tag{70}$$

Thus

$$P(\omega_1|p,q) = \frac{P(p,q,\omega_1)}{\sum_\omega P(p,q,\omega)} = \frac{2K}{n\bar{q}+2K}, \tag{71}$$

which gives (31). For constant μ, we note the similarities between (25) and the first-order condition for single-node networks with m symmetric firms [29].

$$Q = (p - C'(Q))Q'(m-1). \tag{72}$$

By comparing (25) and (72) we can conclude that the first-order solution of a firm in a symmetric two-node network with n firms per node is the same as for a firm in an isolated node with inelastic demand and μn symmetric firms. Thus analytical solutions to (72) are also solutions to (25) when $m = \mu n$. For single node networks, we know that explicit solutions can be derived for symmetric firms facing an inelastic demand and that these solutions are monotonic [5][24][38], which gives us (32). ■

Proof. (Proposition 4) Local net-imports must equal net-demand in every node, so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \\ S_3(p_3) \\ S_4(p_4) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1(\mathbf{p}) \\ t_2(\mathbf{p}) \\ t_3(\mathbf{p}) \end{bmatrix}}_{\mathbf{t}}. \tag{73}$$

Thus

$$\mathbf{A}_{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$\mathbf{E} = \left((\mathbf{A}_{-1})^T\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{74}$$

Each arc i has three congestion states. In the uncongested state we have $\sigma_i = 0$, $\rho_i = 0$ and $t_i \in (-K, K)$. When the arc is congested towards node 4 we have $t_i = K$, $\sigma_i = 0$, and $\rho_i \geq 0$ and when the arc is congested away from node 4 we have $t_i = -K$, $\sigma_i \geq 0$, and $\rho_i = 0$. Altogether there are $3 \times 3 \times 3 = 27$ congestion states. In Appendix C, we use (15) to calculate $P(p, q, \omega)$ for one state ω at a time. The results are summarized in Table 1. Each competitor is assumed to submit a symmetric offer $Q(p)$, so $S_2(p) \equiv S_3(p) \equiv S(p) := nQ(p) - D(p)$. Adding the results in Table 1 yields:

$$\sum_{\omega} P(p, q, \omega) = \frac{6KS^2(\overline{p})}{V_2} + \frac{16K^2 S(\overline{p})}{V_2} + \frac{8K^3}{V_2}. \tag{75}$$

Node 1 is completely integrated with either node 2 or 3 in states ω_{15}, ω_{17}, ω_{26}, ω_{27} and completely integrated with both nodes in state ω_{18}. In the other states node 1 is either isolated or only completely integrated with node 4, which does not have any producers and where demand is inelastic. We have

$$\begin{array}{c} P(p, q, \omega_{15}) + P(p, q, \omega_{17}) + P(p, q, \omega_{26}) + P(p, q, \omega_{27}) + 2P(p, q, \omega_{18}) \\ = \frac{4K^2 S(p)}{V_2} + \frac{4K^2(S(\overline{p}) - S(p))}{V_2} + \frac{4K^2 S(p)}{V_2} + \frac{4K^2(S(\overline{p}) - S(p))}{V_2} + \frac{16K^3}{V_2} = \frac{8K^2 S(\overline{p}) + 16K^3}{V_2}, \end{array} \tag{76}$$

and

$$\begin{array}{c} P(\omega_{15}|p, q) + P(\omega_{17}|p, q) + P(\omega_{26}|p, q) + P(\omega_{27}|p, q) + 2P(\omega_{18}|p, q) \\ = \frac{P(p,q,\omega_{15}) + P(p,q,\omega_{17}) + P(p,q,\omega_{26}) + P(p,q,\omega_{27}) + 2P(p,q,\omega_{18})}{\sum_{\omega} P(p,q,\omega)} = \frac{4KS(\overline{p}) + 8K^2}{3S^2(\overline{p}) + 8KS(\overline{p}) + 4K^2}. \end{array} \tag{77}$$

This gives (41), because demand is inelastic, so $S(\overline{p}) := n\overline{q}$, and

$$\mu = 1 + P(\omega_{15}|p, q) + P(\omega_{17}|p, q) + P(\omega_{26}|p, q) + P(\omega_{27}|p, q) + 2P(\omega_{18}|p, q).$$

It follows from (14), (75) and (76) that

$$\begin{array}{c} Z(p, q) = (p - C'(q)) \left(S'_{1,-g}(p) \left(\frac{6KS^2(\overline{p})}{V_2} + \frac{16K^2 S(\overline{p})}{V_2} \right) + \frac{8K^2 S(\overline{p}) + 16K^3}{V_2} S'(p) \right) \\ -q \frac{2K}{V_2} \left[3S^2(\overline{p}) + 8KS(\overline{p}) + 4K^2 \right]. \end{array}$$

We note that $\frac{\partial Z(p,q)}{\partial q} \leq 0$, so if we find a monotonic stationary solution, then it is an equilibrium. The two explicit equilibrium expressions and monotonicity of these solutions can be established as in the proof of Proposition 3. ■

Appendix C: Calculations for congestion states in star network

In the following we use $\mathbf{A}_B^{\mathcal{F}}(\omega)$, $\mathbf{E}_U^{\mathcal{F}}(\omega)$ and $\mathbf{E}_L^{\mathcal{F}}(\omega)$ to denote submatrices of $\mathbf{A}_B(\omega)$, $\mathbf{E}_U(\omega)$ and $\mathbf{E}_L(\omega)$ corresponding to nodes in the set $\mathcal{F}(\omega)$.

6.0.3 State ω_1

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\mathcal{F}$
ω_1	K	K	K	$\{1\}$	$\{2, 3, 4\}$

Table 1: The 27 congestion states of the star network.

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	$P(p,q,\omega)$
ω_1	K	K	K	0
ω_2	K	K	$-K$	0
ω_3	K	K	$\in(-K,K)$	$\frac{K\left(S^2(\overline{p})-S^2(p)\right)}{V}$
ω_4	K	$-K$	$-K$	0
ω_5	K	$-K$	$\in(-K,K)$	$\frac{K(S(\overline{p})-S(p))^2}{V}$
ω_6	K	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^2(S(\overline{p})-S(p))}{V}$
ω_7	$-K$	K	K	0
ω_8	$-K$	K	$-K$	0
ω_9	$-K$	K	$\in(-K,K)$	$\frac{KS^2(p)}{V}$
ω_{10}	$-K$	$-K$	$-K$	0
ω_{11}	$-K$	$-K$	$\in(-K,K)$	$\frac{KS(p)(2S(\overline{p})-S(p))}{V}$
ω_{12}	$-K$	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^2S(p)}{V}$
ω_{13}	$\in(-K,K)$	K	K	$\frac{2KS^2(p)}{V}$
ω_{14}	$\in(-K,K)$	K	$-K$	$\frac{2KS(p)(S(\overline{p})-S(p))}{V}$
ω_{15}	$\in(-K,K)$	K	$\in(-K,K)$	$\frac{4K^2S(p)}{V}$
ω_{16}	$\in(-K,K)$	$-K$	$-K$	$\frac{2K(S(\overline{p})-S(p))^2}{V}$
ω_{17}	$\in(-K,K)$	$-K$	$\in(-K,K)$	$\frac{4K^2(S(\overline{p})-S(p))}{V}$
ω_{18}	$\in(-K,K)$	$\in(-K,K)$	$\in(-K,K)$	$\frac{8K^3}{V}$
ω_{19}	K	$-K$	K	0
ω_{20}	K	$\in(-K,K)$	K	$\frac{K\left(S^2(\overline{p})-S^2(p)\right)}{V}$
ω_{21}	K	$\in(-K,K)$	$-K$	$\frac{K(S(\overline{p})-S(p))^2}{V}$
ω_{22}	$-K$	$-K$	K	0
ω_{23}	$-K$	$\in(-K,K)$	K	$\frac{KS^2(p)}{V}$
ω_{24}	$-K$	$\in(-K,K)$	$-K$	$\frac{KS(p)(2S(\overline{p})-S(p))}{V}$
ω_{25}	$\in(-K,K)$	$-K$	K	$\frac{2KS(p)(S(\overline{p})-S(p))}{V}$
ω_{26}	$\in(-K,K)$	$\in(-K,K)$	K	$\frac{4K^2S(p)}{V}$
ω_{27}	$\in(-K,K)$	$\in(-K,K)$	$-K$	$\frac{4K^2(S(\overline{p})-S(p))}{V}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F = \emptyset \quad \mathbf{E}_U^F = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{E}_L^F = \emptyset.$$

Thus it follows from (63) that

$$J_F(\omega_1) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} S_2'(p_2) & -S_2'(p_2) & 0 \\ S_3'(p_3) & 0 & -S_3'(p_3) \\ S_4'(p_4) & 0 & 0 \end{bmatrix} \right| = 0,$$

because $S_4'(p_4) = 0$. Now, we have from (15) that

$$P(p, q, \omega_1) = 0.$$

6.0.4 State ω_2

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_2	K	K	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p, q, \omega_2) = 0.$$

6.0.5 State ω_3

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_3	K	K	$\in [-K, K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{E}_U^F = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E}_L^F = \emptyset.$$

Thus it follows from (63) that

$$J_F(\omega_3) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} 0 & S_2'(p_2) & -S_2'(p_2) \\ -1 & S_3'(p_3) & 0 \\ 1 & S_4'(p_4) & 0 \end{bmatrix} \right| = S_3'(p_3) S_2'(p_2),$$

because $S_4'(p_4) = 0$. Now, we have from (15) that

$$\begin{aligned} P(p, q, \omega_3) &= \int_{\rho_1=0}^{\overline{p}-p} \int_{\rho_2=0}^{p+\rho_1} \int_{t_3=-K}^{K} f(\mathbf{A}\mathbf{t} + \mathbf{s}(\mathbf{p}(\pi, \boldsymbol{\rho}, \boldsymbol{\sigma}))) S_3'(p_3) S_2'(p_2)\, dt_3 d\rho_2 d\rho_1 \\ &= \frac{2K}{V_2} \int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1) \int_{\rho_2=0}^{p+\rho_1} S_2'(p+\rho_1-\rho_2)\, d\rho_2 d\rho_1 \\ &= \frac{2K}{V_2} \int_{\rho_1=0}^{\overline{p}-p} S_3'(p+\rho_1) S_2(p+\rho_1)\, d\rho_1 \\ &= \frac{K\left(S^2(\overline{p}) - S^2(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.6 State ω_4

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_4	K	$-K$	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_4)=0.$$

6.0.7 State ω_5

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_5	K	$-K$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F=\begin{bmatrix}0\\-1\\1\end{bmatrix}\quad \mathbf{E}_U^F=\begin{bmatrix}1\\1\\1\end{bmatrix}\quad \mathbf{E}_L^F=\begin{bmatrix}-1\\0\\0\end{bmatrix}.$$

Thus it follows from (63) that

$$J_F(\omega_5)=\left|\frac{\partial\boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\begin{bmatrix}0&S_2'(p_2)&S_2'(p_2)\\-1&S_3'(p_3)&0\\1&S_4'(p_4)&0\end{bmatrix}\right|=S_2'(p_2)\,S_3'(p_3),$$

because $S_4'(p_4)=0$. Now, we have from (15) that

$$\begin{aligned}P(p,q,\omega_5)&=\int_{\rho_1=0}^{\overline{p}-p}\int_{\sigma_2=0}^{\overline{p}-p-\rho_1}\int_{t_3=-K}^{K}f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)S_2'(p_2)\,S_3'(p_3)\,dt_3d\sigma_2d\rho_1\\&=\frac{2K}{V_2}\int_{\rho_1=0}^{\overline{p}-p}S_3'(p+\rho_1)\int_{\sigma_2=0}^{\overline{p}-p-\rho_1}S_2'(p+\rho_1+\sigma_2)\,d\sigma_2d\rho_1\\&=\frac{2K}{V_2}\int_{\rho_1=0}^{\overline{p}-p}S'(p+\rho_1)\left[S_2(\overline{p})-S_2(p+\rho_1)\right]d\rho_1\\&=\frac{K}{V_2}\left[2S(\overline{p})\,S(p+\rho_1)-S^2(p+\rho_1)\right]_0^{\overline{p}-p}\\&=\frac{K\left(S(\overline{p})-S(p)\right)^2}{V_2},\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.0.8 State ω_6

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_6	K	$\in[-K,K]$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F=\begin{bmatrix}-1&0\\0&-1\\1&1\end{bmatrix}\quad \mathbf{E}_U^F=\begin{bmatrix}1\\1\\1\end{bmatrix}\quad \mathbf{E}_L^F=\emptyset.$$

Thus it follows from (63) that

$$J_F(\omega_6) = \left|\frac{\partial \boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right| = \left|\det\begin{bmatrix} -1 & 0 & S_2'(p_2) \\ 0 & -1 & S_3'(p_3) \\ 1 & 1 & S_4'(p_4)\end{bmatrix}\right| = S_2'(p_2)+S_3'(p_3),$$

because $S_4'(p_4) = 0$. Now, we have from (15) that

$$\begin{aligned} P(p,q,\omega_6) &= \int_{\rho_1=0}^{\overline{p}-p}\int_{t_3=-K}^{K}\int_{t_2=-K}^{K} f(\mathbf{A}\mathbf{t}+\mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))(S_2'(p_2)+S_3'(p_3))\,dt_2dt_3d\rho_1 \\ &= \frac{4K^2}{V_2}\int_{\rho_1=0}^{\overline{p}-p} 2S'(p+\rho_1)\,d\rho_1 \\ &= \frac{8K^2(S(\overline{p})-S(p))}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.9 State ω_7

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_7	$-K$	K	K	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_7) = 0.$$

6.0.10 State ω_8

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_8	$-K$	K	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_8) = 0.$$

6.0.11 State ω_9

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_9	$-K$	K	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}^F_B = \begin{bmatrix} 0 \\ -1 \\ 1\end{bmatrix} \quad \mathbf{E}^F_U = \begin{bmatrix} -1 \\ 0 \\ 0\end{bmatrix} \quad \mathbf{E}^F_L = \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix}.$$

Thus it follows from (63) that

$$J_F(\omega_9) = \left|\frac{\partial \boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right| = \left|\det\begin{bmatrix} 0 & -S_2'(p_2) & -S_2'(p_2) \\ -1 & 0 & -S_3'(p_3) \\ 1 & 0 & -S_4'(p_4)\end{bmatrix}\right| = S_3'(p_3)\,S_2'(p_2),$$

because $S_4'(p_4) = 0$. Now, we have from (15) that

$$\begin{aligned} P(p,q,\omega_9) &= \int_{\sigma_1=0}^{p} \int_{\rho_2=0}^{p-\sigma_1} \int_{t_3=-K}^{K} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\, S_3'(p_3)\, S_2'(p_2)\, dt_3 d\rho_2 d\sigma_1 \\ &= \frac{2K}{V_2} \int_{\sigma_1=0}^{p} S'(p-\sigma_1) \int_{\rho_2=0}^{p-\sigma_1} S'(p-\sigma_1-\rho_2)\, d\rho_2 d\sigma_1 \\ &= \frac{K}{V_2} \int_{\sigma_1=0}^{p} 2S'(p-\sigma_1)\, S(p-\sigma_1)\, d\sigma_1 \\ &= \frac{KS^2(p)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.12 State ω_{10}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{10}	$-K$	$-K$	$-K$	$\{1\}$	$\{2,3,4\}$

With similar calculations as for state ω_1, one gets

$$P(p,q,\omega_{10}) = 0.$$

6.0.13 State ω_{11}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{11}	$-K$	$-K$	$\in [-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{E}_U^F = \emptyset \quad \mathbf{E}_L^F = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus it follows from (63) that

$$J_F(\omega_{11}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} 0 & -S_2'(p_2) & S_2'(p_2) \\ -1 & -S_3'(p_3) & 0 \\ 1 & -S_4'(p_4) & 0 \end{bmatrix} \right| = S_2'(p_2)\, S_3'(p_3),$$

because $S_4'(p_4) = 0$. Now, we have from (15) that

$$\begin{aligned} P(p,q,\omega_{11}) &= \int_{\sigma_1=0}^{p} \int_{\sigma_2=0}^{\overline{p}-p+\sigma_1} \int_{t_3=-K}^{K} f(\mathbf{At} + \mathbf{s}(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})))\, S_2'(p_2)\, S_3'(p_3)\, dt_3 d\sigma_2 d\sigma_1 \\ &= \frac{2K}{V_2} \int_{\sigma_1=0}^{p} S'(p-\sigma_1) \int_{\sigma_2=0}^{\overline{p}-p+\sigma_1} S'(p-\sigma_1+\sigma_2)\, d\sigma_2 d\sigma_1 \\ &= \frac{2K}{V_2} \int_{\sigma_1=0}^{p} S'(p-\sigma_1)\left(S(\overline{p}) - S(p-\sigma_1)\right) d\sigma_1 \\ &= \frac{K\left(2S(\overline{p})\, S(p) - S^2(p)\right)}{V_2}, \end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.14 State ω_{12}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{12}	$-K$	$\in[-K,K]$	$\in[-K,K]$	$\{1\}$	$\{2,3,4\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F=\begin{bmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{bmatrix}\quad \mathbf{E}_U^F=\emptyset\quad \mathbf{E}_L^F=\begin{bmatrix}-1\\ -1\\ -1\end{bmatrix}.$$

Thus it follows from (63) that

$$J_F(\omega_{12})=\left|\frac{\partial\boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\begin{bmatrix}-1 & 0 & S_2'(p_2)\\ 0 & -1 & S_3'(p_3)\\ 1 & 1 & S_4'(p_4)\end{bmatrix}\right|=S_3'(p_3)+S_2'(p_2),$$

because $S_4'(p_4)=0$. Now, we have from (15) that

$$\begin{aligned}P(p,q,\omega_{12}) &= \int_{\sigma_1=0}^{p}\int_{t_2=-K}^{K}\int_{t_3=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)\left(S_3'(p_3)+S_2'(p_2)\right)dt_3dt_2d\sigma_1\\ &= \frac{4K^2}{V_2}\int_{\sigma_1=0}^{p}2S'(p-\sigma_1)\,d\sigma_1\\ &= \frac{8K^2S(p)}{V_2},\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.0.15 State ω_{13}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{13}	$\in[-K,K]$	K	K	$\{1,4\}$	$\{2,3\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F=\emptyset\quad \mathbf{E}_U^F=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\quad \mathbf{E}_L^F=\emptyset.$$

Thus it follows from (63) that

$$J_F(\omega_{13})=\left|\frac{\partial\boldsymbol{\varepsilon}_F}{\partial\left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\begin{bmatrix}-S_2'(p_2) & 0\\ 0 & -S_3'(p_3)\end{bmatrix}\right|=S_2'(p_2)\,S_3'(p_3).$$

Now, we have from (15) that

$$\begin{aligned}P(p,q,\omega_{13}) &= \int_{\rho_2=0}^{p}\int_{\rho_3=0}^{p}\int_{t_1=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)S_2'(p_2)\,S_3'(p_3)\,dt_1d\rho_3d\rho_2\\ &= \frac{2K}{V_2}\int_{\rho_2=0}^{p}S'(p-\rho_2)\,d\rho_2\int_{\rho_3=0}^{p}S'(p-\rho_3)\,d\rho_3\\ &= \frac{2KS^2(p)}{V_2}=\frac{2KS^2(p)}{V_2},\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.0.16 State ω_{14}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{14}	$\in [-K,K]$	K	$-K$	$\{1,4\}$	$\{2,3\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F = \emptyset \quad \mathbf{E}_U^F = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \mathbf{E}_L^F = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus it follows from (63) that

$$J_F(\omega_{14}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \det \begin{bmatrix} -S_2'(p_2) & 0 \\ 0 & S_3'(p_3) \end{bmatrix} \right| = S_2'(p_2)\, S_3'(p_3).$$

Now, we have from (15) that

$$\begin{aligned}
P(p,q,\omega_{14}) &= \int_{\rho_2=0}^{p} \int_{\sigma_3=0}^{\overline{p}-p} \int_{t_1=-K}^{K} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(\pi, \boldsymbol{\rho}, \boldsymbol{\sigma})\right)\right) S_2'(p_2)\, S_3'(p_3)\, dt_1 d\sigma_3 d\rho_2 \\
&= \frac{2K}{V_2} \int_{\rho_2=0}^{p} S'(p-\rho_2)\, d\rho_2 \int_{\sigma_3=0}^{\overline{p}-p} S'(p+\sigma_3)\, d\sigma_3 \\
&= \frac{2KS(p)\left(S(\overline{p}) - S(p)\right)}{V_2} \\
&= \frac{2KS(p)\left(S(\overline{p}) - S(p)\right)}{V_2},
\end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.17 State ω_{15}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	F
ω_{15}	$\in [-K,K]$	K	$\in [-K,K]$	$\{1,3,4\}$	$\{2\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^F = \emptyset \quad \mathbf{E}_U^F = [-1] \quad \mathbf{E}_L^F = \emptyset.$$

Thus it follows from (63) that

$$J_F(\omega_{15}) = \left| \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left(\mathbf{t}^F_{\mathbf{B}(\boldsymbol{\omega})}, \boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})}, \boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})} \right)} \right| = \left| \left[-S_2'(p_2) \right] \right| = S_2'(p_2).$$

Now, we have from (15) that

$$\begin{aligned}
P(p,q,\omega_{15}) &= \int_{t_3=-K}^{K} \int_{\rho_2=0}^{p} \int_{t_1=-K}^{K} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(\pi, \boldsymbol{\rho}, \boldsymbol{\sigma})\right)\right) S_2'(p_2)\, dt_1 d\rho_2 dt_3 \\
&= \frac{4K^2}{V_2} \int_{\rho_2=0}^{p} S'(p-\rho_2)\, d\rho_2 \\
&= \frac{4K^2 S(p)}{V_2},
\end{aligned}$$

where we have used that $S_2(p) = S_3(p) = S(p)$.

6.0.18 State ω_{16}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{16}	$\in[-K,K]$	$-K$	$-K$	$\{1,4\}$	$\{2,3\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

Thus it follows from (63) that

$$J_{\digamma}(\omega_{16})=\left|\frac{\partial\boldsymbol{\varepsilon}_{\digamma}}{\partial\left(\mathbf{t}^{\digamma}_{\mathbf{B}(\boldsymbol{\omega})},\boldsymbol{\rho}_{\mathbf{U}(\boldsymbol{\omega})},\boldsymbol{\sigma}_{\mathbf{L}(\boldsymbol{\omega})}\right)}\right|=\left|\det\begin{bmatrix}S_2'(p_2) & 0\\ 0 & S_3'(p_3)\end{bmatrix}\right|=S_2'(p_2)\,S_3'(p_3)\,.$$

Now, we have from (15) that

$$\begin{aligned}P(p,q,\omega_{16}) &= \int_{\sigma_2=0}^{\overline{p}-p}\int_{\sigma_3=0}^{\overline{p}-p}\int_{t_1=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)S_2'(p_2)\,S_3'(p_3)\,dt_1d\sigma_3d\sigma_2\\ &= \frac{2K}{V_2}\int_{\sigma_2=0}^{\overline{p}-p}S'(p+\sigma_2)\,d\sigma_2\int_{\sigma_3=0}^{\overline{p}-p}S'(p+\sigma_3)\,d\sigma_3\\ &= \frac{2K\left(S(\overline{p})-S(p)\right)^2}{V_2},\end{aligned}$$

where we have used that $S_2(p)=S_3(p)=S(p)$.

6.0.19 State ω_{17}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{17}	$\in[-K,K]$	$-K$	$\in[-K,K]$	$\{1,3,4\}$	$\{2\}$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=[0]\,.$$

Thus it follows from (63) that

$$J_{\digamma}(\omega_{17})=0$$

and we have from (15) that

$$P(p,q,\omega_{17})=0.$$

6.0.20 State ω_{18}

State	$t_1(\omega)$	$t_2(\omega)$	$t_3(\omega)$	Ξ	$\digamma$
ω_{18}	$\in[-K,K]$	$\in[-K,K]$	$\in[-K,K]$	$\{1,2,3,4\}$	$\emptyset$

In this state we have from (73) and (74) that:

$$\mathbf{A}_B^{\digamma}=\emptyset \quad \mathbf{E}_U^{\digamma}=\emptyset \quad \mathbf{E}_L^{\digamma}=\emptyset.$$

Now, we have from (15) that

$$P(p,q,\omega_{18})=\int_{t_3=-K}^{K}\int_{t_2=-K}^{K}\int_{t_1=-K}^{K} f\left(\mathbf{At}+\mathbf{s}\left(\mathbf{p}(\pi,\boldsymbol{\rho},\boldsymbol{\sigma})\right)\right)dt_1dt_2dt_3=\frac{8K^3}{V_2}.$$