Risk trading in capacity equilibrium models

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We present a set of power investment models, the class of *risky capacity equilibrium problems*, reflecting different assumptions of perfect and imperfect markets. The models are structured in a unified stochastic Nash game framework. Each model is the concatenation of a model of the short-term market operations (perfect competition or Cournot), with a long-term model of investment behavior (risk neutral and risk averse behavior under different assumptions of risk trading). We prove existence of solutions and report numerical results to illustrate the relevance of market imperfections on welfare and investment behavior. The models can all be formulated as complementarity problems, some of them having an optimization equivalent. The models are constructed and discussed as two stage problems but we show that the extension to multistage is achieved by a change of notation and a standard assumption on multistage risk functions. We also treat a large multistage industrial model to illustrate the computational feasibility of the approach.

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1 Introduction

1.1 Problem statement

Generation capacity expansion models [30] were initially designed for the monopoly electricity industry. The problem was formulated as a minimization of the total annual investment and operations cost of satisfying a given demand decomposed into different time segments. Plant capacity costs are computed by first transforming overnight investment costs into annual equivalents through a standard actualization formula.[1] Total investment cost is then the sum, over all plants, of their annual equivalent investment cost multiplied by their capacity. Operations costs are the sum of fuel costs consumed by capacities in the different time segments of the year. The minimization of the total investment and operations expenses determines these capacities. The so called “merit order” reflects the minimization of operations costs for given capacities: plants are called by order of increasing operations cost until the demand in the time segment is satisfied or curtailment of demand is found necessary. Casting these different ideas within the linear programming capabilities of the time was the significant innovation of these early models that would later be expanded in several directions during the monopoly period of the power sector. Of direct relevance for this paper is the extension of the static yearly model to a succession of periods where demand and cost scenarios can evolve through time. Other extensions, not discussed here, improve the representation of operations (e.g., reliability constraints and probability computation of the operations costs). All those extensions took advantage of numerical progress by moving from linear to non-linear, control and mixed integer programming; they also benefited from the work in decomposition methods.

The extension of these models to encompass other sectors was a first significant change in this evolution. TIMES developed since 2000 by the International Energy Agency’s “Energy Technology System Analysis Programme” is probably the best known of these expanded models today.[2] This deals with several fuels in competition and hence could not be driven by a single optimizing planner. This required a market interpretation that the restructuring of the power and gas sectors also made compulsory. This shift had no computational implications as it derived from fundamental economic principles: the primal-dual optimality conditions of linear programs in activity analysis can be interpreted in terms of perfectly competitive economies [10]; the dual variables of balance equations are interpreted as prices and primal variables become actions of individual agents (producers and consumer) maximizing their profits (for companies) and welfare (for consumers) with respect to prices. This

[1]The economic life of generation plants are much longer than one year, requiring overnight investment costs to be converted into equivalent annual costs (annuities).

[2]See http://iea-etsap.org for an introduction to TIMES and it’s predecessor MARKAL [16].

economic interpretation justified, at least to some extent, the restructuring of the power market: the minimization of operations cost under given capacities and its associated merit order rule can be seen as resulting from a perfectly competitive market implemented through an auction where producers and consumers respectively bid supply and demand curves. This short-term market gives prices that generators will then use to expand capacities as in other industries. To improve things further, financial products can help by providing hedging instruments against the risk of departure from long-term forecasts.

Needless to say the perfectly competitive market is an ideal paradigm and market imperfections pervade reality. Indeed it isn't hard to find an electricity market where consumer prices have been increasing and investment has stalled except those that benefit from subsidies [8]. A rational attitude is to seek some of the causes of the overall failure of the restructuring in market imperfections and to cast them in computational form for assessing them in realistic models. This is admittedly still an academic exercise but it is closer to reality than assuming away the market imperfections that standard economic theory tells us to be aware of. Divergences between real markets and the paradigm of perfect competition can indeed be found in both the short and long-term restructured power markets.

This paper provides a unifying computational framework for mixing both perfect competition and strategic behavior at least within a certain domain. For the short run market, we consider the standard perfect competition and Cournot paradigms. For the long run, we explore different implications of risk. Specifically we replace the standard deterministic scenarios by an event tree of scenarios similar to the one used in stochastic programming or finance theory. Given that event tree we successively consider situations where (i) one or several risk neutral agents invest and consume, (ii) a single risk averse agent decides investment and invest; we show that this situation can be interpreted as a decentralized market with an extensive set of risk trading instruments and (iii) a set of risk averse individual agents that invest, consume and trade risk to mitigate the impact of uncertainty. For reasons explained later in the survey of the literature we do not consider the case where agents exercise market power in the investment stage (strategic withholding of investment) but we refer to the computational literature that considers that problem. Mixing the two short run market imperfections and the three long run market imperfections, we deal with six problems. As we discuss, the computational framework allows for more, but this is left for further research.

1.2 Our approach

We treat these problems in a unifying conceptual and computational framework by formulating them as stochastic Nash equilibrium models. We give existence results for these different models and illustrate the relevance of the analysis via a numerical example, the main results of which are summarized in Table 1, which shows each of the 6 cases corresponding to a choice of risk neutral or risk averse agents, and competitive or Cournot spot markets, and, for the risk averse cases, complete risk markets or no risk trading. As expected the greatest welfare and the greatest investment in power generation capacity both occur in the case of risk neutral perfectly competitive agents; these provide benchmark figures against which we express relative welfare and relative capacity outcomes in all cases. As can be seen from the Table, market imperfections may have dramatic effects on welfare and willingness to invest. Market imperfections can thus be quite important and should be tackled explicitly at policy level.

Risk-attitude	Spot market	Financial trading	Relative welfare	Relative investment	Problem type
Neutral	Competitive	-	100.0	100.0	optimization [a]
Neutral	Cournot	-	89.0	66.6	Nash game
Averse	Competitive	Complete	79.3	90.0	optimization [b]
Averse	Competitive	No risk trading	76.6	87.0	equilibrium
Averse	Cournot	Complete	70.6	60.3	Nash game [c]
Averse	Cournot	No risk trading	69.7	59.0	equilibrium

Table 1: Relative welfare (%), relative capacity installed (%) and problem type for the 6 different cases on an illustrative example

Note that the formulation of the investment problem in a risky environment leads in general an equilibrium problem that cannot be conveniently reformulated as a convex optimization problem. In Table 1, we highlight two situations when the problem is amenable to simpler formulation. For example, it is well know that with risk neutral competitive agents, the investment problem can be solved as a stochastic capacity expansion problem [a]. In this paper, we show that complete financial trading allow a formulation to a special Nash game [c], and that this Nash game can further be simplified to an optimization problem when the market is competitive [b].

Section 1.3 coming after this introduction briefly places our work in the existing literature: we present our models as extensions of former capacity expansion problems that we adapt to the environment of competition and risk prevailing in the industry.

1.3 Literature survey

The literature on generation capacity expansion models is abundant. [50] offers an insightful survey of cost minimization models and associated economic problems developed during the regulation period. [29] presents a general view of capacity expansion problems while issues related to the transition from the regulated monopoly to competition are discussed in [23]. The TIMES model, which is used on a world basis (see reports available from the Energy Technology Systems Analysis Programme), is a state of the art example of cost minimization tools. [6] gives a comprehensive state of the art presentation of computational generation and transmission capacity expansion models including those where investments are made strategically, i.e., investment decisions account for their impact on the subsequence spot market equilibrium. We take stock of that work to discuss aspects of the literature that relate to the themes of the paper.

Uncertainty is today a key feature of the power market. Risk is not a market failure but incomplete risk trading is (also referred as a missing market problem in the economic literature [37]). Imperfect competition whether in the short or long run is also a market failure. Market design is also prone to generate these failures but these are left for further research. All of these distort investment. The extension of the basic cost optimization capacity expansion model to a stochastic model seeking the expected minimal investment and operations cost is the first and obvious step that comes to mind to get into these issues. Chapter 3 of [6] gives an in depth treatment of stochastic generation capacity expansion models from the viewpoint of a social planner.

As already mentioned the move from cost to market models relies on the interpretation of the optimality conditions of the optimal dispatch in terms of a perfectly competitive short-term electricity market. It may be paradoxical that Boiteux, who has always been a fierce advocate of the monopoly in the power industry, was also the first author to have pointed out that relation. The book of Schweppe and his co-authors [48] and the early papers of Hogan ([21, 22] developed this relation between cost and markets into the methodological background of the restructuring of the power industry. [3] summarizes these relations for perfectly competitive generation and transmission markets. Our treatment of perfect competition in the short-term market directly derives from these texts: it assumes a short-term market cleared by a one stage, single settlement, in each hour.

One can immediately note that the primal-dual conditions of the stochastic capacity expansion model can again be interpreted as describing a stochastic equilibrium in a perfectly competitive market provided one assumes that all agents are risk neutral and share the same probabilities of the different possible developments of the market; we mention that TIMES includes this extension [28]. The investment criterion derived from the KKT conditions of the stochastic version of the model retains the usual formulation of the investment criterion provided one replaces the revenue and cost of the deterministic model by their expectations. The short-term market clearing in every scenario is also perfectly competitive. This model will be taken as the basis of our analysis of the literature.

The economic literature was quick to contest the assumption of a perfectly competitive short-term power market and to reveal the exercise of market power. Because our paper deals with computational issues, we skip that literature except for mentioning [38] and more specifically its first two sections that point to the special role played by the so-called Cournot model in the analysis of market power (the bibliography of [38] provides a rich set of relevant economic literature). The Cournot paradigm indeed quickly emerged as a convenient and relevant alternative to perfect competition when there is evidence of imperfect competition. It is not a perfect description of reality (in fact there is no empirically perfect representation of market power) but a reasonable and easy to use approximation for exploring strategic issues at a macro level. It is also related to industry concentration criteria used in competition law and hence offers some welcome links to practice. Besides being conceptually convenient for economists the Cournot model is also attractive for computation: it is based on the same merit order rule as the optimal dispatch that underpins the perfect competition model, only differing from the latter by the fixing of the price. [51, 31, 24, 20] typify treatment in the computational literature. The Cournot model is our reference paradigm for modeling market power in the short term.

Conjectural variations were introduced in 1924 in the economic literature as a generalization of the Cournot model. The notion has perhaps lost its appeal among economists but [7] introduced a version of it computational oligopolistic models for enlarging the scope of imperfect competition in the short-term market. This was largely followed in the computational literature because it could be used as a (somewhat ad hoc) remedy to technical difficulties encountered with sophisticated uses of the Cournot model as discussed next. For the sake of conciseness we do not elaborate further on conjectured supply functions but simply mention that they can be directly embedded in our computational framework.

Notwithstanding its attractiveness when applied to the sole energy market (as in this paper), the Cournot model quickly becomes intractable (both computationally and in terms of its economic interpretation) when extended to more complex situations that also encompass services (such as energy and transmission or balancing). The economic difficulty is to identify reasonable assumptions on how market power affects the different submarkets (e.g., competitive energy market and locally concentrated transmission markets or market power in both energy and transmission or market power in energy only when transmission capacity is ample). The computational challenge is to nest Cournot representations of submarkets (e.g., exercising market power on energy taking into account that it is also exercised on transmission). This leads to so-called Equilibrium Problem subject to Equilibrium Constraints [17]. EPEC problems are intractable and may have multiple solutions that need to be analyzed to identify those that are economically undesirable (e.g., that show and excessive amount of market power). [52] offers an in depth discussion of economic and (some) numerical aspects on a problem of energy and transmission. [46] is just one example of the many papers dealing with EPEC through mixed integer reformulation; its interest is to focus on the analysis of these multiple equilibria. [17] provides an extensive discussion of these questions and many examples of EPEC models.

It appears difficult to bypass the difficulties raised by applying Cournot beyond the single stage, single market model (which justifies resorting to conjectured supply functions): referring again to [38], these authors propose to add a financial market before the physical Cournot settlement to improve the realism of the model. This is indeed empirically justified but it transforms the model into an EPEC with rather bad properties [34]. [38] also proposes to replace the variable price minus cost margin of Cournot by a constant one that should in principle be simpler and more robust in practice. The authors present their proposal for a market with identical producers for which they obtain an explicit solution. The reader can immediately verify that removing the assumption of identical producers leads to an EPEC! Summing up, notwithstanding the difficulty of generalizing the Cournot model (let alone more complex models such as supply function equilibrium models that are not discussed here but are referred to in [38]) to multimarket, it remains a practical way to get a macro assessment of the impact of market power. This is probably what is needed for looking at investment.

One can close this discussion of spot markets by noting that short-term market imperfection can also be the direct (sometimes unavoidable) result of the market design. Except for transmission and the debate on nodal vs. zonal systems, this topic has received much less attention. Leaving aside subjects related to average cost pricing in the regulatory period [19], or in transmission [51], one should mention the work on the two settlement market of [32]. These authors note that the short-term market rarely takes the form of a single fuel cost minimization as assumed here, and in most of the literature, but is implemented through of a sequence of such problems (the two settlement model). They show that the two settlement market can be reformulated as a complementarity problem. [26] also discusses that type of model for linking successive market clearing without going through an EPEC. The reality is that several short-term market imperfections can be formulated through single stage complementarity problems that can directly be embedded in our computational framework. We leave the issue for further research.

Strategic withholding of investment is a real possibility in restructured power markets. Chapter 6 of [6] discusses that problem. Their model is a computational (and hence in principle general) extension of a stylized model discussed analytically in [33]. Problems of this type will always be EPECs: even the simplest representation of market power (the Cournot model) will give that result as soon as embedded in a multistage (here long term market followed by a short term market) problem. Besides the usual computational difficulties accruing from EPEC, [35] shows that these models can turn out to be extremely unstable, with small changes of data leading to discontinuities of the behavior of the market. Based on this experience we do not model market power at the investment level both because of numerical and economic interpretation difficulties. The reader can consult [6, Chapter 6] and references therein for experience with these models.

Strategic behavior can sometimes be difficult to differentiate from prudent conduct in the face of risk [41]. The analysis of prudent conduct in the presence of risk requires moving away from risk neutrality to risk averse for modeling the behavior of agents. Two cases can be considered. One is to return to the optimization capacity expansion model and to replace the expectation total cost of in the objective function by a risk function. The alternative is to stick to the stochastic equilibrium model but to assume that each agent behaves according to a risk function and no longer on the basis of expectation.

Risk optimization is a relatively recent, but already extensively developed, extension of conventional stochastic programming. [49] gives a comprehensive presentation of both the notions of risk functions and their use in optimization. In contrast with the risk neutral market case, the interpretation of a risk averse market (where individual agents are risk averse) model does not directly follow from an analysis of the KKT conditions of the risk optimization model (a risk averse planner). The relation between risk-averse capacity expansion models was first discussed in [13] where the authors analyze the conditions for which the KKT conditions of a risk-averse capacity expansion optimization model can effectively be interpreted as a risk-averse equilibrium. These authors show that these conditions are equivalent to the existence of a "complete financial market". In related work, [41] gives a similar analysis on hydro management. In the meantime a study of risk-averse agents who can influence the "design" of their risky assets and hedge those assets in complete risk markets was studied as a risky equilibrium model in [42], revised to include incomplete risk trading in [43]. The recourse to more general utility functions is an alternative representation to the construction of risk averse stochastic equilibrium model: [27] states such a problem through the KKT conditions of the agent's maximization problems. Similarly [47] states the problem in Nash Equilibrium terms using an exponential utility function. Neither of these papers discuss the relation between their model and a possible optimization problem. Risk aversion in the presence of complete risk markets is our second assumption of long-term market imperfection. Incomplete markets, that is when individual agents are modeled as behaving according to a risk function, with and without financial instruments, are treated in [1].

Our models are stochastic Nash games. It is worth distinguishing these from a substantial literature recently developed on monotone stochastic Nash games (e.g., [25, 44]). In the latter each agent optimizes an objective function which is convex in its decision variable given the decisions of the others, as in our models, under the assumption that the combination of the mappings derived from individual agents' objectives is monotone in the Cartesian product of agents' actions. The incompleteness of risk trading, mentioned as the last long term imperfection, violates this latter assumption.

1.4 Structure of paper

The building blocks of this paper are the classical risk neutral models of capacity expansion for stochastic spot markets that are either competitive or Cournot together with recent work on risky design equilibria [43] for agents whose risk aversion is characterized by coherent risk measures [2]. The last topic is reviewed in section 2 and introduces a risk pricing agent whose role is to set the probability measure that all other (risk trading) agents use to value their risky assets; see Theorem 1.

Section 3 studies capacity expansion for stochastic and competitive spot markets. We combine risk trading with capacity expansion for agents who are competitive in investment, risk trading and production. The main goal of the section is to give an interpretation of a risk averse social planner to the capacity equilibrium problem when the financial market is complete. That is, the *risky competitive capacity equilibrium problem with complete markets* (Definition 3) is equivalent to optimization under risk (Theorem 3). This is a direct extension of the classical reformulation of a risk neutral competitive capacity equilibrium problem as a risk neutral optimization problem (Theorem 2). We contrast the risk neutral, complete and incomplete cases via numerical examples (cf. Table 1).

Section 4 is devoted to capacity expansion for Cournot spot markets, focussing on the *risky Cournot capacity equilibrium problem with complete markets* (Definition 4) in which agents invest in plant capacities and trade risk products prior to stochastic Cournot production. We show in Theorem 4 that this equilibrium problem is equivalent to a pure strategy Nash game in which the producers jointly participate in a classical risk neutral Nash Cournot capacity game with a twist: the probability measure under which all generators evaluate stochastic outcomes is endogenously chosen by the risk pricing agent. We also compare the risk neutral, complete and incomplete cases via numerical examples.

The goal of section 5 is to show that in the case of competitive spot markets, the multistage situation is effectively equivalent to the two stage case. The main interest, therefore, may be in a realistic multistage numerical example that is presented in section 5.1.2. The notation for translation from multistage back to two stages is given in section 5.2 and applied in Theorem 6 to give an optimization interpretation of the *multistage risky competitive capacity equilibrium problem with complete markets*; this is multistage version of Theorem 3.

Section 6 provides sufficient conditions for existence of risky capacity equilibria with *incomplete markets* in both cases of competitive and Cournot spot markets. See Theorems 8 and 9. Section 7 concludes the paper and an Appendix appears after presenting the bibliography to give a myriad of proof details that range from benign but tedious to structurally interesting.

2 Risk trading and stochastic design equilibrium problems

2.1 Notation for uncertainty and coherent risk measures

Later we will see "design" variables $x \in \mathbb{R}^n$ that describe investments into risky assets such as production plants whose future outputs and hence future profit — in fact we will model uncertain cost as negative profit. $\mathbb{R}^n_+$ denotes the set of nonnegative vectors in $\mathbb{R}^n$. The Euclidean inner product will be denoted $u^\top v$ for $u, v \in \mathbb{R}^n$.

The space of uncertain outcomes (costs) is $\mathcal{Z} := \mathbb{R}^K$. That is, uncertainty or randomness or stochasticity is characterized by a number, K, of stochastic scenarios indexed by $\omega \in \Omega := \{1, \ldots, K\}$. We may refer to a member Z of $\mathcal{Z}$ as (Z_ω) or $(Z_\omega)_\omega$ depending on the setting.

A probability density (probability measure) is a nonnegative vector $\Pi \in \mathbb{R}^N$ whose entries sum to 1. The set of probability measures is denoted $\mathcal{P}$. In the risk neutral case we are given $\Pi \in \mathcal{P}$ and evaluate the cost of any $Z \in \mathcal{Z}$ as an expectation or inner product,

$$\mathbb{E}_\Pi[Z] \;:=\; \Pi[Z] \;:=\; \sum_\omega \Pi_\omega Z_\omega.$$

In many instances the stochastic phenomenon has a data source, for example a time series of electricity prices. In that case the data is associated with an empirical probability measure that we call the physical probability measure, also called the real world probability measure, and denoted this by Θ.

For our purposes, coherent risk measures (CRMs) are defined as support functions over sets of probability measures. That is, r is a CRM if and only if for some nonempty, closed and convex set $\mathcal{D}$ of probability measures we have $r(Z) = \max_{\zeta \in \mathcal{D}} \zeta[Z]$. The closed convex set $\mathcal{D}$ associated with the CRM $\max_{\Pi \in \mathcal{D}} \mathbb{E}_\Pi[\cdot]$ is called its *risk set*. This definition of a CRM is equivalent to saying r is real valued function on $\mathbb{R}^K$ that satisfies a set of four axioms; see [2] which initiated the axiomatic approach and the associated equivalence. Occasionally we may refer to one of the axioms that a CRM r must satisfy without reference, so we list them here: Axiom 1, convexity. Axiom 2, monotonicity: $r(Z_1) \le r(Z_2)$ for $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \le Z_2$, where the inequality is taken scenario-wise. Axiom 3, translation invariance: $r(Z + \alpha \mathbb{1}) = r(Z) + \alpha$ for $Z \in \mathcal{Z}$ and $\alpha \in \mathbb{R}$. Axiom 4, positive homogeneity: $\rho(\alpha Z) = \alpha \rho(Z)$ for $Z \in \mathcal{Z}$ and $\alpha > 0$.

It may be convenient for modeling or computation to reformulate a CRM as a convex minimization problem. When the risk set $\mathcal{D}$ is expressed using constraints such as nonlinear inequalities, this can be accomplished by using standard convex duality. For later use we specify a generic minimization description of a CRM,

$$\min_{u} g(u, Z) \text{ subject to } G(u, Z) \in \mathcal{K} \tag{1}$$

where u is a finite dimensional vector, g is a smooth convex function, $\mathcal{K}$ is a closed convex cone in a finite dimensional vector space and G is a smooth function that is convex with respect to $\mathcal{K}$.[3]

Looking ahead, we have an interest in a CRM r_0 whose risk set is formed by intersecting the risk sets of two (or more) other CRMs r_1 and r_2. Suppose r_1 and r_2 have the same minimization representation up to parameterization, i.e., using the same functional forms g and G that are parameterized as

$$g = g(\cdot\,;\gamma_i),\ G = G(\cdot\,;\gamma_i)$$

given scalar or vector parameters γ_1 and γ_2, and same cone $\mathcal{K}$. In the examples to follow it is clear that r_0 may inherit the same representation where its parameter γ_0 is calculated as an elementary function of (γ_1, γ_2). That is, the minimization representation may also be convenient for constructing parameterized families of CRMs.

Example 1. CV@R, [45]. The CV@R, which has several other names including expected shortfall and several other, is a popular risk measure in stochastic optimization. Its risk set is defined by

$$\mathcal{D}_{\text{CV@R}} := \{\Pi \text{ is a probability measure} | \Pi_\omega \le \gamma^{-1}\, \Theta_\omega\}$$

where Θ is physical probability measure. The convex minimization formulation of the CV@R is given by the following Linear Program (LP):

$$r_{\text{CV@R}} = \min_{t,U} \left\{ t + \gamma^{-1}\mathbb{E}_\Theta[U] \quad \text{subject to} \quad \begin{aligned} & U \in \mathbb{R}^K_+, \\ & U - Z + t \in \mathbb{R}^K_+ \end{aligned} \right\}$$

Example 2. Good deal, [5]. The valuation of uncertain payoff is usually approached by two methodologies: arbitrage pricing (based on replication arguments, eg. Black and Scholes formula) and equilibrium theory (based on the exposure to fundamental sources of macroeconomic risk, eg. consumption-based CAPM). [4] introduced the good-deal pricing as a mix of the two, where they impose a valuation that rules out arbitrage opportunities as well as

[3]This is equivalent to convexity of the real-valued function $\lambda^\top G$ for each λ in $\mathcal{K}^\circ := \{\lambda : \lambda^\top d \le 0,\ \text{for all } d \in \mathcal{K}\}$.

assets with too high Sharpe ratio (because at equilibrium those "good-deals" would quickly disappear as investors would immediately grab them). The risk set of the good deal risk measure is given by

$$\mathcal{D}_{GD} := \left\{ \Pi \text{ is a probability measure} \,\middle|\, \mathbb{E}_{\Theta}\left[\left(\frac{\Pi_\omega}{\Theta_\omega} \right)^2 \right] \leq \gamma^2 \right\} \tag{2}$$

where Θ is the physical probability measure. In finance, this expression is also known as the Hansen-Jagannathan bound. The calibration of this risk measure requires one parameter γ linked to a targeted Sharpe ratio: $\gamma^2 = 1 + SR^2$. A value of about 0.5 is usually representative of the US market.

The convex minimization formulation of the good-deal from [11] requires to solve the following Second Order Cone Program (SOCP).

$$\begin{aligned} r_{GD} = \min_{s,t,U} \Big\{ t + \gamma s \text{ subject to } \quad & U \in \mathbb{R}_+^K, \\ & U - Z + t \in \mathbb{R}_+^K, \\ & \left(\sqrt{\Theta}\, U, s \right) \in \mathbb{L}^{K+1} \Big\} \end{aligned} \tag{3}$$

The symbol $\mathbb{L}$ is the Lorentz cone, defined as $(x, y) \in \mathbb{L}^{N+1} := \{(x, y) \in \mathbb{R}^{N+1} : y \geq \|x\|_2\}$

Below we will be concerned with some risk sets that have interiority relative to $\mathcal{P}$. For each of the CV@R and good deal CRMs, the corresponding risk sets have interiority if and only there is a probability measure that strictly satisfies the inequalities defining the risk set.

2.2 Review of stochastic design equilibria and risky design Nash games

We review a noncooperative game of N agents. The first decision of agent i is a vector of design variables x_i that lies in a given decision or strategy set $X_i \subset \mathbb{R}^{n_i}$. We write $\mathcal{I} := \{1, \ldots, N\}$,

$$\begin{aligned} x_{\mathcal{I}} &= \left(x_i\right)_{i \in \mathcal{I}}, & x_{-i} &:= \left(x_j\right)_{i \neq j \in \mathcal{I}}, \\ X_{\mathcal{I}} &:= X_1 \times \ldots \times X_N, & X_{-i} &:= \bigtimes_{\hat{\imath} \in \mathcal{I} \setminus \{i\}} X_{\hat{\imath}}. \end{aligned}$$

The design vector x_i describes agent i's investment (e.g., capacity) in a risky asset Ξ_i (e.g., a cost or negative profit stream from a production plant that operates in an uncertain market), which is also affected by other agents' decisions hence has the form $\Xi_i(x_i, x_{-i}) := (\Xi_{i\omega}(x_i, x_{-i})) \in \mathcal{Z}$.

This noncooperative form allows to model various levels of competition in the market, from competitive agents (i.e. price-taking agents) to strategic ones as in the Cournot paradigm. To that end we introduce some structure on the uncertain costs that fits our capacity expansion situations: Each scenario cost $\Xi_{i\omega}(x_i, x_{-i})$ is the optimal value of a production optimization problem carried out by agent i, e.g., it chooses its (second stage) production quantity y_i given that the plant capacity and operating cost is determined by both (first stage parameters) x_i and x_{-i}. The generic version of this two stage optimization problem is given in the next result which is entirely standard in parametric convex optimization. $T_X(x)$ denotes the tangent cone to the set X at one of its members x.

Lemma 1. *Let X be a closed convex subset of $\mathbb{R}^n$, $f : X \times \mathbb{R}^m \to \mathbb{R}$ be convex and continuous, $g : X \times \mathbb{R}^m \to \mathbb{R}^\ell$ be componentwise convex and continuous, and, for each $x \in X$,*

$$v(x) \;:=\; \min_y \left\{ f(x,y) \;\text{ subject to }\; g(x,y) \leq 0 \right\}$$

be achieved as a minimum. Then, as a function on X, v is convex and continuous. Moreover if X is polyhedral and g is polyhedral then for any $x \in X$ and $d \in T_X(x)$, there exists the directional derivative $v'(x;d)$.

In fact under much weaker conditions on X and g than polyhedrality we can show that v is directionally differentiable at $x \in X_i$ in any direction d in the relative interior of $T_{X_i}(x_i)$.[4] The relevance is that this weaker directional differentiability opens the door to the application of all of the ideas in this paper, from Theorem 1 (below) onwards, to recourse problems involving nonlinear constraints. To keep the exposition focussed and simple we stick with polyhedral feasible sets, i.e., linear constraints, at least with respect to constraints on investments.

As a shorthand, we say v is the *optimal value function on X of a parametric fully convex and linearly constrained program* if the hypotheses of the lemma hold and g is also polyhedral, where "fully" reflects convexity of the optimization objective and polyhedrality of the constraints in (x,y) not just y. (An example of this situation is when both f and g are both polyhedral hence the parametric optimization problem is equivalent to a linear program and the optimal value function is polyhedral convex.) Below we will require $\Xi_{i\omega}(\cdot, x_{-i})$ to be the optimal value function on a polyhedral convex set X_i of a parametric fully convex and linearly constrained program.

In order to assesses the cost of an uncertain outcome, we endow each agent i with a coherent risk measure $r_i : \mathcal{Z} \to \mathbb{R}$, e.g., it values its uncertain cost $\Xi_i(x_i, x_{-i})$ as $r_i(\Xi_i(x_i, x_{-i}))$. This leads to another decision made by each agent i: What basket of financial products, represented in a vector $W_i \in \mathcal{W}$ where $\mathcal{W}$ is a subspace of $\mathcal{Z}$, it should it buy to hedge its risk via $r_i(\Xi_i(x_i, x_{-i}) - W_i)$? This question should account for the cost of financial products, represented by the price of risk P^{r} which is considered to be a vector dual to $\mathcal{W}$, i.e., an

[4]Such conditions are related to the study of stability conditions in optimization such as calmness and constraint qualifications.

element in $\mathcal{W}^*$ so that the cost of W is the Euclidean inner product $P^{\rm r}[W]$. Agent i's optimization with respect to both design and hedging variables is

$$\min_{x_i, W_i} P^{\rm r}[W_i] + r_i\big(\Xi_i(x_i, x_{-i}) - W_i\big) \quad \text{subject to} \quad x_i \in X_i\ ,\ W_i \in \mathcal{W}. \tag{4}$$

The price of risk $P^{\rm r}$ is determined by the equilibrium condition that all trades of financial products balance each other:

$$\sum_{i=1}^{N} W_i = 0. \tag{5}$$

Throughout the paper, we consider the three following situations:

1. the risk neutral (RN) case where all agents value their stochastic cost based on the expectation under a given probability measure (the physical probability measure Θ);

2. the case with complete markets where agents are risk averse and all risks can be traded, i.e., W_i is unconstrained in $\mathcal{Z}$;

3. the case with incomplete markets where agents are risk averse and W_i can only vary in a proper subspace $\mathcal{W}$ of $\mathcal{Z}$.

The risk neutral case is the mostly studied in the literature and it is well known that, for risk neutral agents, modeling risk trading (with endogenous price) is irrelevant to the equilibrium. The reasoning is that risk trading can only have zero premium at an equilibrium, otherwise all agents would take unbounded trading position. With such zero premium, all risk neutral agents are indifferent to trade risk, i.e., the design decision x_i is unchanged (see also the discussion after Definition 1 to follow). This section formulates the generic problem, but then mainly focuses on the case with complete markets. The incomplete case will be further reviewed in section 6.1.

Our basic assumptions underlying risky design equilibrium problems are specified next. For $\Pi \in \mathcal{P}$ we write $\Pi > 0$ to denote $\Pi_\omega > 0$ for all ω. Accordingly we write $\Pi \not> 0$ to denote a probability measure where $\Pi_\omega = 0$ for at least for one ω.

Design Assumptions.

1. X_i is a nonempty polyhedral set in $\mathbb{R}^{n_i}$ for each $i \in \mathcal{I}$.
2. $\Xi_{i\omega}(\cdot, x_{-i})$ is the optimal value function on X_i of a parametric fully convex and linearly constrained program, for each ω and $x_{-i} \in X_{-i}$.

CRM assumptions.

1. Each $r_i : \mathcal{Z} \to \mathbb{R}$ is a CRM with a nonempty closed convex risk set D_i, i.e., $r_i(Z) = \max_{\Pi \in \mathcal{D}_i} \mathbb{E}_\Pi[Z]$ for each $Z \in \mathcal{Z}$. $\mathcal{D}_i > 0$ is called agent i's *risk set.*
2. The *system risk set* $\mathcal{D}_0 := \cap_i \mathcal{D}_i$ is nonempty. We call $r_0 := \sigma_{\mathcal{D}_0}$ the *system CRM.*
3. One of the two conditions below holds:
   (a) Either, each risk set $\mathcal{D}_i$ is polyhedral and convex.
   (b) Or, $\mathcal{D}_0$ has nonempty interior relative to $\mathcal{P}$.
4. $\mathcal{D}_0 > 0$ by which we mean $\Pi > 0$ for each $\Pi \in \mathcal{D}_0$.

In our later numerical examples we will use risk sets of the good deal type. This means that the reader may have to keep in mind 3(b) in the CRM assumptions. For agents who all use good deal risk sets, interiority of $\mathcal{D}_0$ relative to $\mathcal{P}$ is equivalent to having a probability measure that strictly satisfies all the inequalities that are used to define agents' risk sets, see (2).

Definition 1 (Stochastic design equilibrium models).

1. The RN design equilibrium problem *is the system where each agent i seeks a solution x_i of a risk neutral optimization problem,*

$$\min_{x_i} \quad \mathbb{E}_\Pi\big[\Xi_i(x_i, x_{-i})\big] \quad \text{subject to} \quad x_i \in X_i\,. \tag{6}$$

2. The risky design equilibrium problem with complete markets *is the system that combines* (4) *(for all i), where $\mathcal{W} = \mathcal{Z}$, and* (5)*. To be explicit, a tuple $(x^*_{\mathcal{I}}, W^*_{\mathcal{I}}, P^{\mathrm{r},*})$ is a solution of the risky design equilibrium problem if*

(i) For all $i \in \mathcal{I}$, (x^*_i, W^*_i) is a solution of

$$\min_{x_i, W_i} \Big\{ P^{\mathrm{r},*}[W_i] + r_i\big(\Xi_i(x^*_i, x_{-i}) - W_i\big) \quad \text{subject to } x_i \in X_i \Big\} \tag{7}$$

(ii) The financial market clears, i.e., (5) *holds.*

3. The risky design equilibrium problem with incomplete markets *is the system that combines* (4)*, where $\mathcal{W}$ is a proper subspace of $\mathcal{Z}$, and* (5)*.*

Observe that Definition 1 presents a hierarchy of equilibrium formulations from least to most general. The RN design equilibrium model can be recovered from the risky design problem with complete markets, given the probability measure Π, by taking each risk measure $r_i := \mathbb{E}_\Pi$ because then stationarity with respect to W_i in (7) gives $P^{\mathrm{r}} = \Pi$ and (7) collapses to (6). Of course (7) is a special case of (4), so that the complete case is generalized by the incomplete formulation if we allow $\mathcal{W} = \mathcal{Z} = \mathbb{R}^K$ in the latter.

Our main tool in analyzing risky design equilibrium problems with complete markets is to reformulate them as a nearly-risk-neutral game, in Definition 2. (Analysis of the incomplete case via an incomplete risk neutral game is presented in section 6.) The equivalence between the equilibrium and game formulations is given in Theorem 1.

Definition 2 (**Risky design game with complete markets).** *The* risky design or Nash game with complete markets *combines* (6) *(for all i) with*

$$\max_{\Pi} \mathbb{E}_{\Pi}\left[\sum_{i=1}^{N} \Xi_i(x_i, x_{-i})\right] \quad \text{subject to} \quad \Pi \in \mathcal{D}_0. \tag{8}$$

The problem (8) *is attributed to a (system)* risk pricing agent*.*

It is of interest that the game above is "stochastic endogenous" in that the probability measure Π used by each design agent in its risk neutral optimization problem is determined, at equilibrium, by the system risk agent. Note also that the system risk agent's problem, (8), is the evaluation of the system risk measure $r_0 = \sigma_{\mathcal{D}_0}$ at the aggregate (or system) cost $\sum_i \Xi_i(x_i, x_{-i})$.

The following result was developed [43] in the slightly more general case discussed after Lemma 1 above, which would allow for nonlinearity in the parametric fully convex optimization problems underlying each component $\Xi_\omega(x_i, x_{-i})$.

Theorem 1 (**Risky design equilibrium problem with complete markets as a game).** *[43, Corollary 2]*
If the Design Assumptions and parts 1–3 of the CRM Assumptions hold, then

1. *The risky design equilibrium problem with complete markets and the risky Nash game with complete markets are equivalent:* $(x_{\mathcal{I}}, P^{\mathrm{r}})$ *with some* $W_{\mathcal{I}}$ *is a risky design equilibrium with complete markets if and only if* $(x_{\mathcal{I}}, \Pi = P^{\mathrm{r}})$ *solves the risky Nash game with complete markets.*

2. *A risky design equilibrium with complete markets* $(x_{\mathcal{I}}, W_{\mathcal{I}}, P^{\mathrm{r}})$ *is such that* $\Pi = P^{\mathrm{r}}$ *simultaneously solves the CRM evaluation of the system risk agent,* (8)*, and of each other agent:*
$$\max_{\Pi} \mathbb{E}_{\Pi}\big[\Xi_i(x_i, x_{-i}) - W_i\big] \quad \text{subject to} \quad \Pi \in \mathcal{D}_i.$$

The risky Nash game with complete markets is simpler than the corresponding equilibrium problem because it does not need risk trades $W_{\mathcal{I}}$; the design variables $x_{\mathcal{I}}$ form a solution of a risk neutral Nash game, (6), rather than a risk averse game; and the probability measure used in (6) is optimized by the system risk agent rather than a price that is implicit in a market clearing condition.

A final point is that the condition $D_0 > 0$ (part 4 of the CRM Assumptions) is convenient when analyzing two- and multistage decision processes — see discussion after Definitions 3 and 4 in sections 3 and 4.1, respectively — but is inessential to Theorem 1 and to the paper as a whole.

***Remark* 1** (*Aside: The Risky design game with complete markets as a tatonnement heuristic)*.
This simplification provides a useful decomposition that can be used by heuristic based on proximal algorithms, surveyed in some generality in [40], where

- for all i, (6) is replaced by the following proximal operators

$$\mathbf{prox}_{\lambda,i}(v_i) = \arg\min_{x_i} \left(\mathbb{E}_\Pi\left[\Xi_i(x_i, x_{-i}) - W_i \right] + (1/2\lambda)\|x_i - v_i\|_2^2 \quad \text{subject to} \quad x_i \in X_i \right) ,$$

- and (8) is replaced by

$$\mathbf{prox}_{\lambda}(p) = \arg\max_{\Pi} \left(\mathbb{E}_\Pi\left[\sum_{i=1}^{N} \Xi_i(x_i, x_{-i}) \right] + (1/2\lambda)\|\Pi - p\|_2^2 \quad \text{subject to} \quad \Pi \in \mathcal{D}_0 \right) .$$

When, for a given Π, the (risk neutral) game with the N agents given by (6) can be solved efficiently (e.g., when amenable to a single convex optimization problem), this heuristic can be further simplified and potentially lead to faster resolution time.

3 Two stage stochastic competitive capacity equilibria

We study two stage equilibrium problems describing capacity planning for a single commodity, here electricity, which is traded in a future, i.e., uncertain, competitive spot market. The competitive assumption requires that all economic agents participating to the spot market are price-takers, i.e., they don't act strategically to influence prices. Along with risk aversion in making capacity investments we introduce risk trading, i.e., a market for financial products, and elaborate on the three situations described in the previous section, i.e., risk neutral, complete and incomplete cases (see parts 1–3 of Definition 3).

For the risk neutral cases, hence without risk trading, it is well known that the competitive equilibrium is equivalently represented by a risk neutral stochastic program; this classical result is reprised Theorem 2 to follow. Computational frameworks that exploit this include the stochastic TIMES model described in [28].

In the complete case, the equilibrium satisfies the standard axioms of perfect competition and is called a *risky competitive capacity equilibrium with complete markets*. Our main result of this section, Theorem 3, shows that a risky competitive capacity equilibrium problem with complete markets has an optimization interpretation in which a central planner solves a risk averse stochastic program. In particular, it maximizes social welfare where the metric for welfare accommodates all agents' risk aversion attitudes. This reprises a key theme of [13]. The optimization interpretation is that an equilibrium can be determined by solving a convex risk averse optimization problem, i.e., which is considerably more reliable and faster than solving the equilibrium formulation. Beyond the clean and general statement here, which can be applied to any perfectly competitive market, allowing for multiple goods, distributed markets with congestion pricing, etc., our proof technique is also informative as seen in the Appendix. It builds directly results from risk neutral welfare economics and design games (section 2). This bypasses the need to translate equilibrium models into stationary conditions as seen in prior papers.

The third case corresponds to the risky competitive capacity equilibrium with incomplete markets and involves risk averse agents that can only partially trade risk trade, see Definition 3.3. There is no (known) reformulation of this case as a convenient optimization problem, i.e., with complexity similar to that of the convex optimization problem confronting each agent. Here we give some computational results but postpone existence theory to section 6.

3.1 Notation, assumptions and model review of competitive markets

Stage 0 consists of investment decisions in capacities of production plants that use different technologies, and also in financial products (contracts for hedging risk). These investments jointly anticipate an uncertain second stage which is a competitive (i.e. agents are price-takers) spot market for electricity.

We model just two agents, one electricity generating company or genco, indexed by $i = 1$, and one electricity retailer representing elastic demand, $i = 2$. All functions—including investment cost, production cost, negative utility and risk measures— will be taken to be convex and real valued (everywhere). As detailed next, we use n_i to denote the dimension of the investment variables, n the number of technologies or production plant types, and a single commodity representing electricity.

Competitive Spot Market Assumptions

1. For $i = 1, 2$ and $\omega \in \Omega$, $I_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and $C_\omega : \mathbb{R}^n \to \mathbb{R}$ are convex cost functions for investment and production, respectively, and $U_\omega : \mathbb{R} \to \mathbb{R}$ is a concave function for utility of consumption.
2. For $i = 1, 2$, X_i is a nonempty convex polyhedral set in $\mathbb{R}^{n_i}$.

3. (a) For $i = 1, 2$ and all ω, we are given $A_{i\omega} \in \mathbb{R}^{\ell_i \times n_i}$, $B_{1\omega} \in \mathbb{R}^{\ell_1 \times n}$ and $B_{2\omega} \in \mathbb{R}^{\ell_2}$, and $b_{i\omega} \in \mathbb{R}^{\ell_i}$ that define the second stage (recourse) constraints for $Y_\omega \in \mathbb{R}^n$ and $Q_\omega \in \mathbb{R}$, respectively,

$$A_{1\omega} x_1 + B_{1\omega} Y_\omega + b_{1\omega} \leq 0, \tag{9}$$

$$A_{2\omega} x_2 + B_{2\omega} Q_\omega + b_{2\omega} \leq 0. \tag{10}$$

   (b) The second stage decision sets $\{Y_\omega : (9) \text{ holds}\}$ and $\{Q_\omega : (10) \text{ holds}\}$ are nonempty[5] for each $x_i \in X_i$ and ω.

   (c) There exists a competitive spot market equilibrium for each ω (see further notation in section 3.1.2 to follow).

The simplest setting is linear investment costs, production costs and utility functions; $X_1 = \mathbb{R}^n_+$, $X_2 = \mathbb{R}_+$; and recourse constraints $0 \leq y \leq x_1$ and $0 \leq q \leq x_2$ for all ω. The recourse formulation (9) and (10) has the additional richness that the dimension of the investment variable n_i need not match the dimension of the recourse variable (n for generation or 1 for retail), and the constraints may vary with ω. The assumption of existence in condition 3(c) is inherent in the discussion[6].

Also, in line with discussion after Lemma 1 in section 2.2, there is no fundamental reason to restrict constraints to be of the linear programming type. The underlying theory, both for risky design problems and for competitive (and likewise strategic) equilibria, allows for X_i to be nonpolyhedral and the recourse constraints (9)–(10) to be nonlinear. To admit non-linearity we would also have to include technical conditions to ensure the associated optimal value functions (to follow in section 3.1.2) are sufficiently well behaved. For tractability of presentation here, we will restrict ourselves to the straightforward polyhedral case.

The basic setting above, with only one producer, one retailer or aggregate consumer and one commodity, is enough to establish the equilibrium template that combines physical and financial investment as well as subsequent production decisions. In fact any number of agents, any number of commodities and any number of stages can be accommodated. As described in the previous literature survey, these kinds of equilibrium models have been used to inform companies and policy makers, particularly in the energy industry, about long term trends, i.e., how capacity will grow or shrink in tandem with changes in demand and prices.

[5]In stochastic programming, this property is known is known as relatively complete recourse.

[6]It is also far from stringent, e.g., follows immediately under either boundedness of feasible sets or a more natural economic condition that marginal cost, at high enough levels of production, exceeds the marginal utility, at corresponding levels of consumption.

3.1.1 Stage 0: Real and financial investments

In stage 0 there are no physical assets for production or retailing. The genco, designated as agent 1, will make a capital expenditure decision in each of n electricity production technologies by selecting capacity $x_1 \in \mathbb{R}^n_+$ of power plants in those technologies at a total investment cost of $I_1(x_1)$. Similarly the retailer, agent 2, will choose its service capacity as the scalar $x_2 \geq 0$ at an investment cost of $I_2(x_2)$.

This stage 0 investment translates to an uncertain outcome in stage 1, in fact a stochastic cost of production or a stochastic utility of consumption, reflecting an uncertain commodity market. The genco and retailer are risk averse; each puts a value on uncertain assets using a coherent risk measure as we will explain later. They each choose investment levels to minimize their costs, or equivalently to maximize their profits, though we will write each investment problem as a minimization of cost net of revenue.

The genco and retail assess uncertain outcomes $Z \in \mathcal{Z}$ using coherent risk measures, $r_1(Z)$ and $r_2(Z)$ respectively. Each r_i is represented by a *risk set* which is a nonempty closed convex set of probability measures $\mathcal{D}_i$ such that $r_i(Z) = \sigma_{\mathcal{D}_i}(Z)$ as discussed in section 2. Also as discussed there, agents engage in risk trading to manage their exposure to risk, e.g., if agent 1 has uncertain operational cost $Z_1 \in \mathcal{Z}$ then it may hedge this by buying forward contracts, represented by $W_1 \in \mathcal{Z}$, in order to change its exposure from $r_1(Z_1)$ to $r_1(Z_1 - W_1)$. We will elaborate on that after modeling the uncertain return of an investment x_i, which comes next.

3.1.2 Stage 1: Competitive equilibrium in each scenario of the spot market

Given n plant capacities bundled as x_1, a spot market scenario ω (in Ω) and a price of power $P_\omega \geq 0$, the genco wants to operate as efficiently as possible. Its production decision is a vector $Y_\omega \in \mathbb{R}^n$ that is feasible with respect to capacity and has a production cost of $C_\omega(Y_\omega)$. The amount of power offered by the genco to the spot market will be the inner product $e^\top Y_\omega = \sum_{j=1}^n (Y_\omega)_j$ where $e = (1, \ldots, 1) \in \mathbb{R}^n$ and $(Y_\omega)_j$ is the jth component of Y_ω; and the associated revenue is $P_\omega \, e^\top Y_\omega$. Thus in spot scenario ω, the genco will maximize its operating profit or minimize its net operating cost,

$$V_{1\omega}(x_1, P_\omega) \;:=\; \min_{Y_\omega} \left\{ C_\omega(Y_\omega) - P_\omega \, e^\top Y_\omega \;\text{ subject to}\,(9) \right\}. \tag{11}$$

Similarly the retailer, given its service capacity which is a scalar x_2, acts nonstrategically in that it purchases a feasible quantity Q_ω of electricity at unit price P_ω and sells it on to consumers at the value of their utility $U_\omega(Q_\omega)$. In scenario ω the retailer's minimum net operating cost is

$$V_{2\omega}(x_2, P_\omega) \;:=\; \min_{Q_\omega} \left\{ P_\omega Q_\omega - U_\omega(Q_\omega) \;\text{ subject to}\,(10) \right\}. \tag{12}$$

Given scenario ω and capacity (x_1, x_2), we say that P_ω *clears the (spot or commodity) market, or is an equilibrium (spot or commodity) price,* if $V_{1\omega}(x_1, P_\omega)$ and $V_{2\omega}(x_2, P_\omega)$ are achieved as minima by some Y_ω and Q_ω, respectively, and the market clearing complementarity condition holds,

$$0 \leq e^\top Y_\omega - Q_\omega \quad \perp \quad P_\omega \geq 0, \tag{13}$$

where $\perp$ indicates orthogonality, i.e., $(e^\top Y_\omega - Q_\omega) P_\omega = 0$. The equilibrium problem (11)–(13) represents perfect competition in the spot market in that no firm is aware of the actions of others or considers the effect of its quantity decision on the spot price. By a *competitive spot equilibrium*, in scenario ω, we mean any $(Y_\omega, Q_\omega, P_\omega)$ that solves (11)–(13).

A fundamental result of welfare economics says that the spot market in each scenario ω is equivalent to a system optimization problem:

$$V_{0\omega}(x_1, x_2) \;\; := \;\; \min_{Y_\omega, Q_\omega} \left\{ C_\omega(Y_\omega) - U_\omega(Q_\omega) \text{ subject to (9), (10) and } Q_\omega \leq e^\top Y_\omega \right\}. \tag{14}$$

Indeed, the equilibrium spot price can be recovered as the dual or Karush-Kuhn-Tucker (KKT) multiplier, whose sign is chosen to be nonnegative, for the constraint $Q_\omega \leq e^\top Y_\omega$.

We observe that $V_{i\omega}(x_i, p)$ and $V_0(x_1, x_2)$ are examples of optimal value functions of parametric fully convex and linearly constrained optimization problems, cf. Lemma 1 in section 2.2, which sets us up to later apply the results on risky design games to the uncertain cost of a competitive agent or the social planner.

Next we highlight some standard results from welfare economics that shows how these value functions, and the spot price, are related at equilibrium.

Proposition 1. *If the Competitive Spot Market Assumptions hold then, for any capacities* $(x_1, x_2) \in X_1 \times X_2$ *and spot scenario* $\omega \in \Omega$,

1. *the (nonempty) set of spot market equilibrium quantities* (Q_ω, Y_ω) *corresponds to the set of optimal solutions of the system optimization problem* (14)*;*

2. *for any solution* (Y_ω, Q_ω) *of* (14)*, the associated (nonempty) set of nonnegative KKT multipliers* P_ω *corresponding to the constraint* $Q_\omega \leq e^\top Y_\omega$ *is the set of equilibrium spot prices; and*

3. $V_{0\omega}(x_1, x_2) = V_{1\omega}(x_1, p) + V_{2\omega}(x_2, P_\omega)$ *for any equilibrium spot price* P_ω.

3.2 Stochastic competitive capacity equilibrium problems

The risk neutral equilibrium model posed as part 1 of the next definition is entirely standard. It is presented here to set notation and foster comparison between the risk neutral and, nonstandard, risk averse equilibrium models with risk trading in parts 2 and 3, where the closest precedents for the complete case include [13] and [9, 1] for partial risk trading. The multistage version of Definition 3 will be presented in section 5.

Recall that $\Omega = \{1, \dots, K\}$ and $\mathcal{Z} = \mathbb{R}^K$. We will use vector notation to represent uncertain outcomes over scenarios $\omega \in \Omega$,

$$Y := (Y_\omega)_\omega \in \mathbb{R}^{n\times K},\ Q := (Q_\omega)_\omega \in \mathbb{R}^K,\ P := (P_\omega)_\omega \in \mathbb{R}^K.$$

Definition 3 (**Stochastic competitive capacity equilibrium models).** *Let the Competitive Spot Market Assumptions hold.*

1. *Let* Π *be a probability measure. In the* (two stage) RN competitive capacity equilibrium problem *we seek:*

   (a) *an investment* x_i *and a stochastic quantity* Y *or* Q*, corresponding to* $i = 1$ *or* 2*, which solve the risk neutral capacity problem,*

$$\begin{aligned} &\min_{x_1,Y} \quad I_1(x_1) + \mathbb{E}_\Pi\big[C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega\big] \quad \text{subject to } x_1 \in X_1, (9) \textit{ for all } \omega, \\ &\min_{x_2,Q} \quad I_2(x_2) + \mathbb{E}_\Pi\big[P_\omega Q_\omega - U_\omega(Q_\omega)\big] \quad \text{subject to } x_2 \in X_2, (10) \textit{ for all } \omega; \end{aligned} \tag{15}$$

   (technically, we further impose that Y_ω*,* Q_ω *solve* (11), (12) *for all* ω *to accommodate for the possibility that* $\Pi \not> 0$*);*

   (b) *a spot price* P_ω *such that the spot market clears* (13) *respectively, for all* ω*.*

2. *Let* $r_i : \mathcal{Z} \to \mathbb{R}$ *(later, a CRM) for* $i = 1, 2$*. In the* (two stage) risky competitive capacity equilibrium problem with complete markets *we seek*

   (a) *an investment* x_i*, stochastic quantity* Y *or* Q*, and risk trade* W_i *that, for* $i = 1, 2$*, solve the risky capacity problem,*

$$\begin{aligned} \min_{x_1,Y,W_1} \quad & I_1(x_1) + P^{\mathbf{r}}[W_1] + r_1\big(C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega - W_{1\omega}\big) \\ \text{subject to} \quad & x_1 \in X_1, (9) \textit{ for all } \omega, \\ \min_{x_2,Q,W_2} \quad & I_2(x_2) + P^{\mathbf{r}}[W_2] + r_2\big(P_\omega Q_\omega - U_\omega(Q_\omega) - W_{2\omega}\big) \\ \text{subject to} \quad & x_2 \in X_2, (10) \textit{ for all } \omega; \end{aligned} \tag{16}$$

   (as in the risk neutral case, we also impose that Y_ω*,* Q_ω *solve* (11), (12)*);*

   (b) *a spot price* P_ω *clearing the spot market* (13) *respectively, for all* ω*;*

(c) a price of risk $P^{\mathbf{r}}$ that clears the risk market,

$$W_1 + W_2 = 0. \tag{17}$$

3. *The* (two stage) risky competitive capacity equilibrium problem with incomplete markets *has the same format (a)–(c) as the equilibrium problem with complete markets while restricting risk trades to lie in a proper subspace $\mathcal{W}$ of $\mathcal{Z}$, i.e.,* (16) *becomes*

$$\begin{array}{ll} \min\limits_{x_1,Y,W_1} & I_1(x_1) + P^{\mathbf{r}}[W_1] + r_1\big(C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega - W_{1\omega}\big) \\ \text{subject to} & x_1 \in X_1, (9) \textit{ for all } \omega, W_1 \in \mathcal{W}, \\ \min\limits_{x_2,Q,W_2} & I_2(x_2) + P^{\mathbf{r}}[W_2] + r_2\big(P_\omega Q_\omega - U_\omega(Q_\omega) - W_{2\omega}\big) \\ \text{subject to} & x_2 \in X_2, (10) \textit{ for all } \omega, W_2 \in \mathcal{W}. \end{array} \tag{18}$$

We give some remarks to clarify the notation in the above definition. First, any parameter that is not visible under a max or min is deemed by the optimizing agent to be fixed and exogenous. In (15) for example, agent i takes the probability measure Π and the stochastic commodity price P as given and does not anticipate that their decision x_i has any impact on those parameters or x_{-i}. Second, the appearance of ω as a subscript indicates an associated stochastic element indexed by $\omega \in \Omega$. For example $r_1(C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega - W_{1\omega})$ is the value of CRM r_1 evaluated at the risky cost vector $(C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega)_\omega - W_1 \in \mathcal{Z}$. Third, in the sequel we will assume that $\Pi > 0$ or $\mathcal{D}_0 > 0$ in the risk neutral or risk averse cases, respectively. Then the scenario-wise equilibrium conditions that Y_ω, Q_ω solve (11), (12) follow from the prior stochastic optimization conditions, hence do not need to be checked separately.

It is also worth noting that Definition 3 presents is hierarchy of equilibrium formulations from least to most general. The RN competitive capacity equilibrium can be recovered from the risky case, given the probability measure Π, by taking each risk measure $r_i := \mathbb{E}_\Pi$ because then stationarity with respect to W_i in (16) gives $P^{\mathbf{r}} = \Pi$ and (16) collapses to (15). The complete case (16) is clearly related to the incomplete case (61) in that both formulations would coincide if we allowed $\mathcal{W} = \mathcal{Z} = \mathbb{R}^K$ in the latter.

3.3 Two stage RN competitive capacity equilibrium

A classical result of welfare economics interprets the RN competitive capacity equilibrium problem as a central planning problem via two stage risk neutral optimization. The latter system problem is to either minimize net cost, as in (19) below, or maximize welfare which is the negative of net cost. We also give a corollary which formulates the system optimization problem as a single stage stochastic program using the system spot value function (14); its statement and proof, though well known, will be helpful templates for subsequent analysis of the risky case.

Theorem 2 (**RN competitive capacity equilibrium problem as risk neutral optimization problem).** *If the Competitive Spot Market Assumptions hold and* $\Pi > 0$*, then* (x_1, Y, x_2, Q, P) *is an RN competitive capacity equilibrium if and only if* (x_1, Y, x_2, Q) *solves the risk neutral system capacity problem*

$$\begin{aligned} \min_{x_1, Y, x_2, Q} \quad & I_1(x_1) + I_2(x_2) + \mathbb{E}_\Pi\big[C_\omega(Y_\omega) - U_\omega(Q_\omega)\big] \\ \text{subject to} \quad & x_1 \in X_1,\ x_2 \in X_2, (9), (10),\ Q_\omega \le e^\top Y_\omega \text{ for all } \omega, \end{aligned} \tag{19}$$

and P *is the nonnegative KKT multiplier corresponding to the final constraint.*

Corollary 1. *In the setting of Theorem 2,* (x_1, x_2) *with some* (Y, Q, P) *is an RN competitive capacity equilibrium if and only if* (x_1, x_2) *solves*

$$\begin{aligned} \min_{x_1, x_2} \quad & I_1(x_1) + I_2(x_2) + \mathbb{E}_\Pi\big[V_{0\omega}(x_1, x_2)\big] \\ \text{subject to} \quad & x_1 \in X_1,\ x_2 \in X_2. \end{aligned} \tag{20}$$

Proof: Let (x_1, Y, x_2, Q, P) be an RN competitive capacity equilibrium. Then (x_1, Y, x_2, Q) solves (19), from Theorem 2. Also since Y_ω, Q_ω, P_ω form a competitive equilibrium in spot market scenario ω,

$$C_\omega(Y_\omega) - U_\omega(Q_\omega) \;=\; V_{1\omega}(x_1, P_\omega) + V_{2\omega}(x_1, P_\omega) \;=\; V_{0\omega}(x_1, x_2) \tag{21}$$

where the second equality is from part 3 of Proposition 1. Thus (19) reduces to the problem (20), indeed the latter must be solved by (x_1, x_2).

For the converse let (x_1, x_2) solve (20). Proposition 1 provides a spot market equilibrium Y_ω, Q_ω, P_ω for each ω, hence a feasible solution for (19). The objective values of these two problems also match via (21). Since $\Pi > 0$ then scenario-wise separability of (19) means that all feasible points have scenario-wise equilibrium quantities Y_ω, Q_ω. This coincidence of objective values and feasible sets gives optimality of (x_1, Y, x_2, Q) for (19). ■

3.3.1 Illustrative example of RN competitive capacity equilibrium

We present a small example with a producer that can invest in one technology. This plant has a annualized capital expenditure of $I = 90$ €/kW and operating cost of $C = 60$ €/MWh. The year is represented by a single segment with a duration of $\tau = 8760$ hours. This producer is risk neutral and solves

$$\min_{x,Y} Ix + \tau\, \mathbb{E}_\Theta\big[(C - P_\omega)Y_\omega\big] \text{ subject to } 0 \le Y_\omega \le x. \tag{22}$$

In this time segment, the consumer has a quadratic utility function $U_\omega(Q_\omega) = A_\omega Q_\omega - \frac{B}{2}Q_\omega^2$ and solves

$$\min_{Q} \tau\, \mathbb{E}_\Theta\Big[P_\omega Q_\omega - A_\omega Q_\omega + \frac{B}{2}Q_\omega^2\Big]. \tag{23}$$

The quadratic term B of the utility is constant across scenarios and equal to 1 €/MWh^2 while the linear term A_ω is the only random parameter in this example. We consider five scenarios ($K = 5$, $\mathcal{Z} = \mathbb{R}^5$) in the second-stage with an equal physical probability of 20%, i.e., $\Theta_\omega = 0.2$ for all ω. Its value in the different scenario is noted in Table 2.

ω	scen 1	scen 2	scen 3	scen 4	scen5
A [€/MWh]	300	350	400	450	500

Table 2: The value of the linear term of utility function across scenarios

The risk neutral competitive equilibrium was solved as the equivalent system minimization (or welfare maximization by taking the negative of the objective), cf., Corollary 1,

$$\min_{Q,x,Y} Ix + \tau \mathbb{E}_\Theta \left[CY_\omega - A_\omega Q_\omega + \frac{B}{2} Q_\omega^2 \right] \quad \text{subject to} \quad Q_\omega = Y_\omega,\ 0 \leq Y_\omega \leq x\,. \tag{24}$$

The optimal investment is $x = 389$ MW. We report in Table 3 the value of the electricity consumption and the price in each scenario, as well as their expectation.

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	240	290	340	389	389	329.6
P [€/MWh]	60.0	60.0	60.0	60.7	110.7	70.3
Investment margin [€/kW]	-90	-90	-90	-84	354	0

Table 3: Risk neutral competitive equilibrium solution

When the utility is quadratic and the consumer is price-taker, the equilibrium solution satisfies the usual linear relationship for price,

$$P_\omega = A_\omega - B_\omega Q_\omega\ , \tag{25}$$

also known as the linear demand function. The equilibrium production Y_ω of the producer in the second stage is given by the following optimality condition:

$$0 \leq Y_\omega \perp \mu_\omega + C - P_\omega \geq 0\,, \tag{26}$$

where μ is the dual variable associated with the capacity constraint $Y_\omega \leq x$ and has the interpretation of a scarcity margin, i.e., a margin made the technology when it operates at full capacity. Otherwise, given the optimality condition (26), the price equals to the cost of the technology, as in scen 1, scen 2 and scen 3. In this table we also report the investment margin which stands for the net profit of investing one unit in the technology. It is computed as $(\tau(P_\omega - C)Y_\omega - Ix)/x$. In the competitive case, this investment margin is solely due to scarcity margin and can also be computed as $\mu_\omega - I$. The producer of the plant is risk neutral and the expected investment margin is 0 €/kW.

The total welfare is given by the expected profit of the producer and the expected surplus of the consumer. The expected profit of the producer is the negative of the optimal value of (22): it is equal to zero at equilibrium as the producer is price-taker and the technology has a linear cost function. The expected surplus of the consumer is the negative of the optimal value of (23). At equilibrium the consumer surplus for each scenario is given by

$$V_{2\omega} = \tau\left(-(A_\omega - BQ_\omega)Q_\omega + A_\omega Q_\omega - \frac{B}{2}Q_\omega^2\right) = \tau\frac{B}{2}Q_\omega^2 . \tag{27}$$

Its expected value is equal to 490.9 M€.

3.4 Reformulating the risky competitive capacity equilibrium problem with complete markets as system optimization under risk

Our first main result, Theorem 3 below, explains how to formulate the risky capacity equilibrium problem with complete markets as optimization under risk. In other words, we show that a risk averse centralized planning formulation of long term capacity risk averse investment, with risk trading, has an economic or market interpretation. The first version of this result is due to [13] where it is derived for an electricity capacity equilibrium model with inelastic demand and unserved or shed load that is priced at an exogenous value of lost load, rather than elastic demand as we do here. A related formulation and result in the multistage case is provided by [41] to analyze the impact of risk aversion on hydro power planning under complete risk trading.

The proofs of related results in [13, 41] are via stationary conditions. We make a rather different approach by viewing Theorem 3 as the fusion of two fundamental results. The first is Theorem 1 (section 2) which characterizes risky design equilibria with complete markets as a combination of a risk neutral Nash game with a risk pricing optimization problem. The second is the classical result Theorem 2 or equivalently Corollary 1, which reformulates an RN competitive capacity equilibrium problem as a risk neutral optimization problem. These results allow us to work directly with equilibrium models instead of translating to and manipulating stationary conditions. See section 8.1 in the Appendix for details.

Theorem 3 (**Risky competitive capacity equilibrium problem with complete markets as risk averse optimization problem).** *If the CRM Assumptions and the Competitive Spot Market Assumptions hold, then*

1. *(x_1, Y, x_2, Q, P) with some $(W_1, W_2, P^{\mathbf{r}})$ is a risky competitive capacity equilibrium with complete markets if and only if (x_1, Y, x_2, Q) solves the two stage risk averse optimization problem,*

$$\begin{array}{ll} \min\limits_{x_1,Y,x_2,Q} & I_1(x_1) + I_2(x_2) + r_0\big(C_\omega(Y_\omega) - U_\omega(Q_\omega)\big) \\ \text{subject to} & x_1 \in X_1,\, x_2 \in X_2, (9), (10),\, Q_\omega \le e^\top Y_\omega \text{ for all } \omega; \end{array} \tag{28}$$

and P is the nonnegative KKT multiplier corresponding to the last constraint.

2. *At a risky competitive capacity equilibrium with complete markets, $\Pi = P^{\mathbf{r}}$ simultaneously solves the CRM evaluations*

$$\begin{array}{lll} \max\limits_{\Pi} \mathbb{E}_\Pi\big[C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega - W_{1\omega}\big] & \text{subject to} & \Pi \in \mathcal{D}_1, \\ \max\limits_{\Pi} \mathbb{E}_\Pi\big[P_\omega Q_\omega - U_\omega(Q_\omega) - W_{2\omega}\big] & \text{subject to} & \Pi \in \mathcal{D}_2, \\ \max\limits_{\Pi} \mathbb{E}_\Pi\big[C_\omega(Y_\omega) - U_\omega(Q_\omega)\big] & \text{subject to} & \Pi \in \mathcal{D}_0. \end{array}$$

Corollary 2. *In the situation of Theorem 3, (x_1, x_2) with some $(Y, Q, P, W_1, W_2, P^{\mathbf{r}})$ is a risky competitive capacity equilibrium with complete markets if and only if (x_1, x_2) solves the risk averse system capacity problem*

$$\begin{array}{ll} \min\limits_{x_1,x_2} & I_1(x_1) + I_2(x_2) + r_0\big(V_{0\omega}(x_1, x_2)\big) \\ \text{subject to} & x_1 \in X_1,\, x_2 \in X_2. \end{array} \tag{29}$$

Remark* 2** *(on Theorem 3).**
First, (28) and (29) are equivalent to maximizing welfare. Moreover these problems are equivalent to linear programs when each C_ω and U_ω are piecewise linear and X_1, X_2 and $\mathcal{D}_0$ are polyhedral.

Second, the optimization formulations above suggests some simple sufficient conditions for existence of equilibria. If design strategy sets X_i are bounded, e.g., if there is a budget constraint on x_i, then a solution exists because (28) has a continuous objective and its feasible set would be nonempty and compact. Alternatively we note that coercivity, hence existence of solutions, occurs under the Competitive Capacity Boundedness assumption which given in section 6.2 and holds naturally as it is argued in the Appendix.

Finally, it is straightforward [43] to recover the equilibrium price of risk P^{r} and risk trades $W_{\mathcal{I}}$ given the equilibrium quantities $x_{\mathcal{I}}$ and spot price P: $P^{\mathbf{r}}$ can be recovered, as noted in Theorem 1, by evaluating $r_0(V_{0\omega}(x_1, x_2)$, i.e., it is the optimal solution Π of

$$\max_{\Pi \in \mathcal{D}_0} \mathbb{E}_\Pi\big[V_{0\omega}(x_1, x_2)\big];$$

while $W_{\mathcal{I}}$ is any solution of

$$\min_{W_{\mathcal{I}}} \sum_i r_i\big(V_{i\omega}(x_i, P) - W_{i\omega}\big) \text{ subject to } \sum_i W_i = 0.$$

In fact P^{r} is also the Lagrange multiplier for the last constraint.

3.4.1 Illustrative example of risky competitive capacity equilibrium with complete markets

We now consider the case of risk-averse agents which operate as price takers in production (second stage) and engage in complete trading of risk at the same time as making their capacity decisions (first stage). The risk aversion of the consumer and the producer is modeled by the good-deal risk measure calibrated on a same Sharpe ratio of 0.5 (i.e. $\gamma = \sqrt{1+0.5^2}$). The producer can buy financial contracts at a market price of risk $P^{\mathbf{r}}[W_1]$ that modifies its profit by $+W_1$, where W_1 can be chosen as any member of $\mathcal{Z}$:

$$\min_{x,Y,W_1} Ix + P^{\mathrm{r}}[W_1] + r_{GD}\big(\tau(C - P_\omega)Y_\omega - W_{1\omega}\big) \text{ subject to } 0 \le Y_\omega \le x \tag{30}$$

The consumer also hedges optimally its risk by taking positions in the financial contracts W_2:

$$\min_{Q,W_2} P^{\mathrm{r}}[W_c] + r_{GD}\big(\tau(P_\omega Q_\omega - A_\omega Q_\omega + \frac{B}{2}Q_\omega^2) - W_{2\omega}\big) \tag{31}$$

The perfectly competitive equilibrium is obtained by combining (30) and (31) with the clearing conditions on energy $Q_\omega = Y_\omega$ and on the financial products $W_1 + W_2 = 0$. It can be solved by the equivalent risk-averse optimization, cf. Corollary 2,

$$\begin{aligned} \min_{t,U,x,Y,Q} \quad & t + Ix + \gamma\sqrt{\mathbb{E}_\Theta[U_\omega^2]} \\ \text{subject to} \quad & U_\omega \ge 0, \\ & U_\omega - CY_\omega - A_\omega Q_\omega + \frac{B}{2}Q_\omega^2 + t \ge 0 \\ & Q_\omega = Y_\omega \\ & 0 \le Y_\omega \le x \end{aligned} \tag{32}$$

The optimal investment is x = 349 MW. The electricity consumption, the price and the system probability measure (the risk probability measure is the same for every agents when risk markets are complete) is given in Table 4.

The electricity price makes the investment profitable in only two scenario (40% probability) but risk trading permits to efficiently hedge this risk. The expected margin of the investment is positive and equal to 142 €/kW, but its risk-adjusted value (=(0.334+0.2757+

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	240	290	340	349	349	313.5
P [€/MWh]	60.0	60.0	60.0	101.3	151.3	86.5
Investment margin [€/kW]	-90	-90	-90	271	709	142
Π	0.334	0.2757	0.2066	0.1302	0.0536	0.2

Table 4: risky competitive capacity equilibrium solution with complete markets

$0.2066) \times -90 + 0.1302 \times 271 + 0.0536 \times 709 = 0)$ is equal to zero: the equilibrium value for a risk-averse competitive producer.

The total welfare is equal to 388.3 M€ and is the sum of the risk-adjusted profit of the producer (objective value of (30)) and the risk-adjusted surplus of the consumer (objective value of (31)). Again, the risk-adjusted profit of a price taking producer is equal to 0 as the technology has a linear cost function.

3.5 Risky competitive capacity equilibrium with incomplete markets

We redevelop the previous example, from section 3.4.1, to give two cases of an incomplete financial market. In the first setting, there are no financial products to hedge the investment. This is equivalent to setting $\mathcal{W} = \{0\}$ though the model we use in computation omits risk trading altogether. It is justified by observations that current financial markets do not provide long-term hedging possibilities for capacities in that contracts become illiquid for a maturity beyond four years [18]. In the second setting, agents can trade one kind of option for hedging the outcome of the low demand scenario ($\mathcal{W}$ is a 1-dimensional subspace of $\mathcal{Z}$).

The inability to trade all risks among agents is a market imperfection. The problem is not amenable to a single optimization problem. Instead we reformulate the equilibrium problem as a complementarity problem, via KKT conditions of its optimization subproblems, and apply the PATH solver [15].

3.5.1 Illustrative example of risky competitive capacity equilibrium with incomplete markets: No risk trading

In the first setting there exist no financial contracts. The total optimal investment of the producer is $x = 339$ MW. The equilibrium solution is reported on Table 5.

When no risk trading is possible, there is no single risk measure or probability measure that describes all agents' behavior at equilibrium. In particular, the probability measure that

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	240	290	339	339	339	309.4
P [€/MWh]	60	60	60.6	110.6	160.6	90.4
Investment margin [€/kW]	-90	-90	-85	353	791	176
Π_1 producer	0.2756	0.2756	0.2740	0.1497	0.0252	0.2
Π_2 consumer	0.3712	0.2574	0.1238	0.1238	0.1238	0.2

Table 5: Equilibrium solution for the competitive equilibrium without any risk trading

is used by each agent to evaluate its CRM at equilibrium may be different, as can be seen by the values of Π_1 and Π_2 in Table 5. The producer cannot fully hedge its investment and requires a higher expected margin of 176 €/kW (note that its risk-adjusted value is again 0 €/kW). The welfare drops to 375.8 M€, which is equal to the risk-adjusted consumer surplus.

3.5.2 Illustrative example of risky competitive capacity equilibrium with incomplete markets: One financial option

We complement the previous case by allowing the two agents (the producer and the consumer) to trade one financial option that partly hedge the uncertainty in demand. This represents a market where demand is correlated to weather and there exists a financial market for weather derivatives (e.g., heating or cooling degree days options traded on the Chicago Mercantile Exchange). The payoff of this option is

	scen 1	scen 2	scen 3	scen 4	scen 5
Option pay off [€]	100	50	0	0	0

Table 6: The payoff of the option

At equilibrium the total capacity increases to x = 348 MW. The equilibrium solution is given in Table 7. One sees the effect of the option on the risk probability measure of the agents (closer on scen 1 and scen 2). This security efficiently hedges producer risk as he invests to a value closed to the case with complete markets. Also the expected net margin for investing reduces to 144 €/kW. The total welfare increases compare to the previous case (no risk trading) to 387 M€.

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
q [MWh]	240	290	340	348	348	313.2
P [€/MWh]	60	60	60	101.7	151.7	86.7
Investment margin [€/kW]	-90	-90	-90	275	713	144
Π producers	0.3352	0.2713	0.2074	0.1358	0.0502	0.2
Π consumer	0.3176	0.3066	0.1986	0.0886	0.0886	0.2

Table 7: Solution for the competitive equilibrium with one financial option.

4 Two stage risky Cournot capacity equilibria with risk trading

This section is the Cournot analog of competitive case developed in section 3. We consider $N-1$ producers and a consumer. The producers engage in both capacity investment and risk trading in the context of a stochastic Cournot spot market, i.e., producers optimally choose their spot quantity to strategically influence prices. The consumer engages in risk trading: its representation is simpler than in the competitive case (no investment in capacity) to make sure that its reaction function in the spot market can easily be included in the producers optimization (differentiable demand function). Modeling risk trading also on the consumer side is economically relevant, he is the natural counterparty for risk sharing.

Cournot capacity equilibria are well studied in the deterministic and risk neutral cases, see, for example, the references provided in [38] and a selection of computational Cournot models represented by [51, 31, 24, 20]

We define the (two stage) stochastic Cournot capacity equilibrium problem in Definition 4, which covers three situations regarding risk: the risk neutral situation, the complete risk trading and the incomplete risk trading. Our main result, Theorem 4, shows that the risky Cournot capacity equilibrium problem with complete markets, in the sense of Definition 2.2, is equivalent to a risky Nash game with complete markets which combines the RN Cournot capacity equilibrium of Definition 4.1 with a risk pricing problem, (44). This immediately leads to sufficient conditions for existence of complete Cournot capacity equilibria based on Nash's classical work [36].

We illustrate the various Cournot capacity equilibrium models numerically, the risk neutral, complete and incomplete cases. Existence theory for the last of these is developed in section 6.

4.1 Notation, assumptions and model review of Cournot spot markets

Stage 0 consists of investment decisions in capacities of production plants that use different technologies, and also in financial products in a risk market. These investments jointly anticipate an uncertain second stage which is a Cournot spot market for electricity. On the supply side we model the investment, in n generation technologies, and production decisions of $N-1$ electricity firms. For the sake of simplicity, demand is only represented by a single consumer agent, the N^{th} agent in the model, whose utility function leads to an elastic demand curve, i.e., the market price of power is a known function of the quantity consumed such that price decreases as quantity increases.

The basic template that we give below, involving convex operational and investment decisions, can be directly extended to accommodate any number of commodities and any convex constraints on production. We list our basic assumptions:

Cournot Convexity Assumptions For $i \in \mathcal{I} = \{1, \ldots, N-1\}$ and $\omega \in \Omega$:

1. $I_i : \mathbb{R}^n \to \mathbb{R}$ and $C_{i\omega} : \mathbb{R}^n \to \mathbb{R}$ are convex cost functions for investment and production, respectively.
2. The commodity price in the spot market scenario ω is a function $p^{\text{C}}_{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ that is directionally differentiable[7] such that the revenue function $q \mapsto p^{\text{C}}_{\omega}(q + q_0)q$ is concave[8] on $\mathbb{R}_+$ for any $q_0 \geq 0$.
3. For each ω, there is a Cournot spot market equilibrium.

To discuss producer i's production decision, $i \in \mathcal{I}$, fix its plant capacity $x_i \geq 0$. Its production in any spot market scenario is constrained by capacity, $0 \leq y_i \leq x_i$, and the total power it offers to the market is $e^{\top} y_i$. In spot scenario ω, the price of power will be $p^{\text{C}}_{\omega}(e^{\top} y_i + q_{-i})$ where q_{-i} denotes the *residual supply*, i.e., the total amount of power generated by other agents,

$$q_{-i} \;:=\; e^{\top} \sum_{\hat{\imath} \in \mathcal{I} \setminus \{i\}} y_{\hat{\imath}}.$$

Residual supply is known by each producer but taken as fixed, i.e., no generator anticipates how its actions will influence others. We write the objective function[9] faced by agent i as

$$R_{i\omega}(y_i, y_{-i}) \;:=\; C_{i\omega}(y_i) - p^{\text{C}}_{\omega}(e^{\top} y_i + q_{-i}) e^{\top} y_i, \tag{33}$$

[7] Directional differentiability at $q = 0$ means directionally differentiable in the direction $dq = 1$.

[8] Concavity of p^{C}_{ω} is sufficient for concavity of revenue.

[9] This notation slightly abuses economic meaning by writing $V^{\text{C}}_{i\omega}$ as a function of $y_{-i} := \left(y_j\right)_{i \neq i}$ instead of q_{-i}, for simplicity.

to show that it acts strategically in choosing y_i: it is aware of the effect of its quantity on price. Its production decision is a solution $Y_{i\omega} = y_i$ of

$$V^{\mathrm{C}}_{i\omega}(x_i, y_{-i}) \;:=\; \min_y \big\{ R_{i\omega}(y, y_{-i}) \;\text{subject to}\; y \in [0, x_i] \big\}. \tag{34}$$

Given $x_{\mathcal{I}}$, (34) specifies a standard Cournot game in production quantities $y_{\mathcal{I}}$ for the spot scenario ω.

Observe that $V^{\mathrm{C}}_{i\omega}(x_i, y_{-i})$ is an example of an optimal value function of a parametric fully convex and linearly constrained optimization problem. So later we can apply the results on risky design games (section 2.2) to the uncertain cost of a Cournot producer.

To include risk trading from the demand as well as the supply side of the market, we specify an electricity consumer. In a Cournot market, the consumer is a nonstrategic agent who does not invest in capacity *per se* and is a quantity-taker. Its utility in spot market scenario ω is

$$U^{C}_{\omega}(Q_\omega) = \int_0^{Q_\omega} p^{\mathrm{C}}_{\omega}(q)dq\;.$$

In other words, the marginal value of consuming the "last" unit Q_ω is $p^{\mathrm{C}}_{\omega}(Q_\omega)$. This demand curve (reaction function) is included in each producer's problem, such that the quantity consumed match with the production $Y_{\mathcal{I}\omega} := (Y_{i\omega})_{i\in\mathcal{I}}$, i.e., $Q_\omega = e^\top \sum_{i\in\mathcal{I}} Y_{i\omega}$. At the Cournot equilibrium price $p^{\mathrm{C}}(Q_\omega)$, the consumer agent obtains a surplus given by

$$V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega}) \;:=\; p^{\mathrm{C}}(Q_\omega)Q_\omega - U^{C}_{\omega}(Q_\omega). \tag{35}$$

This value is entirely decided by producers' actions $Y_{\mathcal{I}\omega}$. In fact although not stated as such, $U^{C}_{\omega}(Q_\omega)$ can be seen to be a degenerate kind of optimal value function of a parametric fully convex and linearly constrained optimization problem, allowing the theory of risky design games to be used with respect to the uncertain cost of the consumer.

4.2 Two stage stochastic Cournot capacity equilibrium problems

The risk neutral two stage Cournot game in part 1 of the next definition is entirely standard, e.g., as embedded in [24]. The risk averse Cournot equilibrium models in part 2 and 3, which involve risk trading, are nonstandard and the contributions here are new, building on risky Nash games as reviewed in section 2.

Definition 4 (Stochastic Cournot capacity equilibrium models).
Let the Cournot Convexity Assumptions hold and $X_i := \mathbb{R}^n_+$ for each i and ω.

1. *Let* Π *be a probability measure. In the* (two stage) RN Cournot capacity game *we seek*

(a) an investment x_i and stochastic quantity Y_i that, for each $i \in \mathcal{I}$, solve the risk neutral capacity problem

$$\begin{array}{rl} \min\limits_{x_i, Y_i} & I_i(x_i) + \mathbb{E}_\Pi\big[R_{i\omega}(Y_{i\omega}, Y_{-i\omega})\big] \\ \text{subject to} & x_i \in X_i,\ 0 \le Y_{i\omega} \le x_i \textit{ for all } \omega, \end{array} \tag{36}$$

(for technical reasons[10]*, we further impose that $Y_{i\omega}$ solves* (34)*. Note also that the spot price P_ω and the consumer optimization $V^{\mathrm{C}}_{N\omega}$ are not explicitly written in the formulation as they are embedded in functions of $Y_{\mathcal{I}\omega}$.) ;*

2. *Let $r_i : \mathcal{Z} \to \mathbb{R}$ (later, a CRM) for $i \in 1, ..., N$. In the* (two stage) risky Cournot capacity equilibrium problem with complete markets *we seek*

   (a) an investment x_i, stochastic quantity Y_i and risk trade $W_i \in \mathcal{Z}$ that, for each $i \in \mathcal{I}$, solve the risky capacity problem

$$\begin{array}{rl} \min\limits_{x_i, Y_i, W_i} & I_i(x_i) + P^{\mathbf{r}}[W_i] + r_i\big(R_{i\omega}(Y_{i\omega}, Y_{-i\omega}) - W_{i\omega}\big) \\ \text{subject to} & x_i \in X_i,\ 0 \le Y_{i\omega} \le x_i \textit{ for all } \omega, \end{array} \tag{37}$$

   (technically, the $Y_{i\omega}$ solves (34) *for each ω)*

   (b) the consumer risk trade $W_N \in \mathcal{Z}$ that solves the risky hedging problem

$$\min_{W_N} \quad P^{\mathbf{r}}[W_N] + r_N\big(V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega}) - W_{N\omega}\big), \tag{38}$$

   (c) a price of risk $P^{\mathbf{r}}$ that clears the risk market,

$$\sum_{i=1}^{N} W_i \ = \ 0. \tag{39}$$

3. *The* (two stage) risky Cournot capacity equilibrium problem with incomplete markets *has the same format (a)–(c) as the complete case while restricting risk trades to lie in a proper subspace $\mathcal{W}$ of $\mathcal{Z}$, i.e.,* (37) *becomes for each $i \in \mathcal{I}$*

$$\begin{array}{rl} \min\limits_{x_i, Y_i, W_i} & I_i(x_i) + P^{\mathbf{r}}[W_i] + r_i\big(R_{i\omega}(Y_{i\omega}, Y_{-i\omega}) - W_{i\omega}\big) \\ \text{subject to} & x_i \in X_i,\ 0 \le Y_{i\omega} \le x_i \textit{ for all } \omega,\ W_i \in \mathcal{W}, \end{array} \tag{40}$$

   and for the consumer agent (38) *becomes*

$$\begin{array}{rl} \min\limits_{W_N} & P^{\mathbf{r}}[W_N] + r_N\big(V^{\mathrm{C}}_{N\omega}(e^\top Y_{\mathcal{I}\omega}) - W_{N\omega}\big), \\ \text{subject to} & W_N \in \mathcal{W} \end{array} \tag{41}$$

[10] As for the competitive case, this accommodates for the possibility that $\Pi \not\succ 0$.

Note the hierarchy in the three Cournot capacity equilibrium models above: The risk neutral case is a special case of the risky case, while the risky case is generalized by the incomplete case in that if we could choose $\mathcal{W}$ to be $\mathcal{Z}$, then (40) we would become (37).

In the sequel we will assume that $\Pi \succ 0$ or $\mathcal{D}_0 \succ 0$ in the risk neutral or risk averse cases, respectively. Then stochastic optimization of production Y_i in (36), (37) or (40) yields optimality of $Y_{i\omega}$ for (34) in each scenario ω, i.e., condition (b) is superfluous in each of the three equilibrium models above. This claim is obvious in the risk neutral case, due to separability in $Y_{i\omega}$, and will be shown later for the risky cases.

4.3 RN Cournot capacity equilibrium

RN Cournot capacity equilibrium problems are a well known type of pure strategy Nash game, and standard game theoretic existence results [14] apply. We only transcribe the specific result for linear demand function when the game can be solved by an optimization problem.

***Remark* 3** (*Cournot with Linear demand function)*.
For the specific linear demand function, i.e., $p_\omega^C(q) = a_\omega^C - b_\omega^C q$, the KKT system of the risk neutral capacity equilibrium problem is integrable (cf. [14]). This property allows to derive an associated convex optimization problem to solve equilibrium problem. This is well known for the deterministic Cournot and transposes directly for the risk neutral case,

$$\begin{array}{ll} \min\limits_{x_\mathcal{I}, Y_\mathcal{I}, Q} & \left(\sum_{i\in\mathcal{I}} I_i(x_i) + \mathbb{E}_\Pi\left[C_{i\omega}(Y_i) + \frac{b_\omega^C}{2}\left(Y_{i\omega}^T Y_{i\omega}\right)\right]\right) - \mathbb{E}_\Pi\left[a_\omega^C Q_\omega - \frac{b_\omega^C}{2} Q_\omega^2\right] \\ \text{subject to} & x_i \in X_i \text{ for all } i, 0 \le Y_{i\omega} \le x_i \text{ for all } i \text{ and } \omega, \\ & e^T Y_{\mathcal{I}\omega} = Q_\omega \text{ for all } \omega. \end{array} \tag{42}$$

The objective function in problem (42) is purely technical and does not have any economical interpretation. As we will show in the next section, this “trick” is only holds in the risk neutral case (and for a linear demand function) but does not work when considering risk-averse agents.

4.3.1 Illustrative example of RN Cournot capacity equilibrium

We analyze a symmetric RN Cournot game with two ($|\mathcal{I}| = 2$) symmetric producers. Each producer acts strategically on its production knowing its effect on the price, $p_\omega^C(y) = A_\omega - By$. The first producer solves

$$\min_{x_1, Y_1} Ix_1 + \tau\mathbb{E}_\Theta\big[\big(C - A_\omega + B(Y_{1,\omega} + Y_{2,\omega})\big)Y_{1,\omega}\big] \quad \text{subject to} \quad 0 \le Y_{1,\omega} \le x_1 \ ,$$

where I, C, A, B and Θ are given in Example 3.3.1. In particular, this Cournot example and the prior perfect competition example are related by having the market price here equivalent to the marginal value of utility in the competitive case. The optimality condition for this producer is given by

$$0 \leq Y_{1,\omega} \perp \mu_{1,\omega} + C - A_\omega + B(2Y_{1,\omega} + Y_{2,\omega}) \geq 0 ,$$

where $\mu_{1,\omega}$ is the dual variable associated to the capacity constraints. In the case of symmetric producers ($Y_{1,\omega} = Y_{2,\omega} = Q_\omega/2$), this expression becomes

$$0 \leq Y_{1,\omega} \perp \mu_{1,\omega} + C - A_\omega + \frac{3B}{2} Q_\omega \geq 0 .$$

This can be rewritten as $0 \leq Y_{1,\omega} \perp \mu_{1,\omega} + C - P_\omega + \frac{B}{2} Q_\omega \geq 0$. The term $\frac{B}{2} Q_\omega$ represents the strategic premium (in €/MWh) of the producer: she only produces when the price is higher than its operating cost plus this value.

As we assume a linear demand function, the risk neutral duopoly equilibrium can be solved by the equivalent optimization problem, see (42), which in this symmetric duopoly case reduces to

$$\min_{Q,x,Y} \; Ix + \tau \mathbb{E}_\Theta \Big[CY_\omega - A_\omega Q_\omega + \frac{3B}{4} Q_\omega^2 \Big] \text{, subject to } Q_\omega = Y_\omega \text{ and } 0 \leq Y_\omega \leq x. \qquad (43)$$

The total optimal investment is $x = 259$ MW. The electricity consumption, the price and the investment margin are reported in Table 8. We breakdown the investment margin in two

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\,]$
q [MWh]	160	193	226	259	259	219.6
P [€/MWh]	140	156.6	173.3	190.5	240.5	179.8
Investment margin [€/kW]	343	542	778	1052	1490	841
- strategic margin [€/kW]	433	632	869	1136	1136	841
- risk margin [€/kW]	-90	-90	-90	-82	356	0

Table 8: Risk neutral duopoly equilibrium solution

part: the strategic margin and the risk margin. The strategic margin is due to the strategic behavior of the duopoly and is equal to $(\frac{B}{2} Q_\omega) \times (\frac{\tau Q_\omega}{x}) = \tau B Q_\omega^2 / (2x)$. The risk margin is the other source of profit for an investment i.e. $(P_\omega - \frac{B}{2} Q_\omega) \times (\frac{\tau Q_\omega}{x}) - I$. The margin due to risk is similar to the risk neutral competitive case (cf. Table 3), and its expectation is also equal to zero as agents are risk neutral. The positive expected net margin of the producers come only from their strategic behavior.

Compared to the competitive case, the welfare decreased to 436.1 M€. The symmetric producers competing á la Cournot have a positive expected profit equal in total to 217.1 M€, while the expected surplus of the consumer drops to 219 M€.

4.4 Reformulating the risky Cournot capacity equilibrium problem with complete markets as a Cournot design game with complete markets

The main result of this section, below, reformulates a Cournot risky capacity equilibrium problem via Theorem 1, from section 2 on risky design equilibrium problems. This converts the equilibrium problem into a risky Nash game with complete markets, comprising a RN Cournot capacity game (36), which seeks an equilibrium $(x_{\mathcal{I}}, Y_{\mathcal{I}})$ given a probability measure Π, with a risk pricing agent who determines Π:

$$\max_{\Pi} \mathbb{E}_{\Pi}\Big[\sum_{i\in\mathcal{I}} R_{i\omega}(Y_{i\omega}, Y_{-i\omega}) + V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega})\Big] \quad \text{subject to} \quad \Pi \in \mathcal{D}_0. \tag{44}$$

This objective of the risk pricing agent in (44) is the sum of agents spot profit (or surplus) and is the same expression as in the competitive case with complete markets, as

$$\begin{aligned}
&\sum_{i\in\mathcal{I}} R_{i\omega}(Y_{i\omega}, Y_{-i\omega}) + V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega}) \\
=\;& \sum_{i\in\mathcal{I}} \Big(C_{i\omega}(Y_i) - p^{\mathrm{C}}_{\omega}(Q_\omega) Y_{i\omega} \Big) \;+\; p^{\mathrm{C}}(Q_\omega) Q_\omega - U^{C}_{\omega}(Q_\omega) \\
=\;& \sum_{i\in\mathcal{I}} C_{i\omega}(Y_{i\omega}) \;-\; U^{C}_{\omega}(Q_\omega)
\end{aligned} \tag{45}$$

We call the combination of (36), (44) a *risky Cournot design game with complete markets* and a solution, a *risky Cournot design equilibrium with complete markets.*

Theorem 4 (**Risky Cournot capacity equilibrium with complete markets as a game**). *If the CRM Assumptions and the Cournot Convexity Assumptions hold, then*

1. $(x_{\mathcal{I}}, Y_{\mathcal{I}}, P^{\mathbf{r}})$*, with some* $(W_{\mathcal{I}}, W_N)$*, forms a risky Cournot capacity equilibrium with complete markets if and only if* $(x_{\mathcal{I}}, Y_{\mathcal{I}}, \Pi = P^{\mathbf{r}})$ *is a risky Cournot design equilibrium with complete markets.*

2. *At a risky Cournot design equilibrium with complete markets,* $\Pi = P^{\mathbf{r}}$ *simultaneously solves the CRM evaluations for the system risk agent,* (44)*, for producer agents* $i \in \mathcal{I}$*,*

$$\max_{\Pi} \mathbb{E}_{\Pi}\big[R_{i\omega}(Y_{i\omega}, Y_{-i\omega}) - W_{i\omega}\big] \quad \text{subject to} \quad \Pi \in \mathcal{D}_i.$$

and for the consumer agent N*,*

$$\max_{\Pi} \mathbb{E}_{\Pi}\big[V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega}) - W_{N\omega}\big] \quad \text{subject to} \quad \Pi \in \mathcal{D}_N.$$

Proof: Part 1. Suppose the tuple $(x_{\mathcal{I}}, Y_{\mathcal{I}}, \Pi = P^{\mathbf{r}})$ solves the risky Cournot design game with complete markets, (36), (44). This is equivalent, from part 1 of Theorem 1, to existence of $W_{\mathcal{I}}$ such that $(x_{\mathcal{I}}, Y_{\mathcal{I}}, W_{\mathcal{I}}, W_N, \Pi = P^{\mathbf{r}})$ solves conditions (a) and (c) of Definition 4.2. It only remains to show that $Y_{i\omega}$ solves (34), which is condition (b) of Definition 4.2. This holds by combining $\Pi_\omega > 0$ with separability of (36) in $Y_{i\omega}$.

Part 2 is given by part 2 of Theorem 1. ■

We end this section by giving sufficient conditions for existence of a risky Cournot capacity equilibrium with incomplete markets.

Cournot Cost Growth Assumption

1. When $x_i \in X_i$, $I_i(x_i) \to \infty$ as $\|x_i\| \to \infty$.
2. For each ω,
   (a) when $x_i, y_i \in \mathbb{R}^n_+$ with $y_i \leq x_i$, $I_i(x_i) + C_{i\omega}(y_i) - p^{\mathrm{C}}_\omega(0) e^\top y_i \to \infty$ as $\|(x_i, y_i)\|_\infty \to \infty$, and
   (b) $p^{\mathrm{C}}_\omega(q)$ is decreasing in $q \geq 0$.

The conditions in this assumption are economically innocuous. For instance part 2(a) has some generality in that it applies to technologies with either high investment cost and low operating cost or *vice versa*. In Europe, since 2010, the former category would include renewable like solar and wind energy, and the latter would be more relevant to traditional thermal generation.

The Cournot Cost Growth Assumption implies coercivity of (36), i.e., the objective function is unbounded if $\|(x_i, Y_i)\| \to \infty$; indeed the lower level sets and hence optimal solution set is nonempty, convex, closed and bounded uniformly with respect to any probability measure Π. This means that the proof of the next existence result is standard. Indeed if we incorporate an appropriate bound on the feasible sets of (36), it is a corollary of Nash's classical result [36]; we omit tedious details.

Corollary 3. *In the situation of Theorem 4, if either the Cournot Cost Growth Assumption holds or each strategy set X_i is bounded, then there exists a risky Cournot capacity equilibrium with complete markets.*

Remark **4** (*Risky Cournot capacity equilibrium with complete markets and a linear demand function)*.
In the specific case of linear demand function, the Cournot equilibrium model with complete markets is not amenable to a single optimization program. It can only be simplified to a

Nash game with two agents: a risk neutral Cournot system agent solving the risk neutral equilibrium problem in (42), and a system risk pricing agent solving

$$\begin{array}{ll} \min\limits_{\Pi} & \mathbb{E}_{\Pi}\left[\sum_{i\in\mathcal{I}} C_{i\omega}(Y_{i\omega}) - a^C_{\omega}(Q_{\omega}) + \frac{b^C_{\omega}}{2}(Q_{\omega})^2\right] \\ \text{subject to} & \Pi \in \mathcal{D}_0. \end{array} \tag{46}$$

This game is not amenable to a single optimization, as the KKT system is not integrable. Compare to the competitive case, the objective of the risk neutral agent in (42) differs from the welfare function in (46).

4.4.1 Illustrative example of risky capacity Cournot equilibrium with complete markets

We elaborate on the risk neutral example, in section 4.3.1, by analyzing the Cournot equilibrium problem with two symmetric risk averse producers who make capacity investments and engage in complete risk trading, also with the consumer (the third agent in the setting). In this setting, the two producers are acting strategically on the electricity market in the second stage but they remain price takers on the financial market. The first producer solves

$$\begin{array}{ll} \min\limits_{x_1,Y_1,W_1} & Ix_1 + P^{\mathbf{r}}[W_1] + \ r_{GD}\Big(\tau\big(C - A_{\omega} + B(Y_{1,\omega} + Y_{2,\omega})\big)Y_{1,\omega} - W_{1\omega}\Big) \\ \text{subject to} & 0 \le Y_{1,\omega} \le x_1 \text{ for all } \omega \end{array} \tag{47}$$

The problem of the second producer is symmetric (i.e. swapping index 1 and 2). The consumer optimally hedges its spot surplus by solving

$$\min_{W_3} \ P^{\mathbf{r}}[W_3] + \ r_{GD}\Big(- \frac{\tau B}{2}(Y_{1,\omega} + Y_{2,\omega})^2 - W_{3\omega}\Big) \tag{48}$$

As shown in remark 4, the problem can at best be simplified to the following Nash game:

$$\begin{array}{l} \min\limits_{Q,x,Y} \ Ix + \tau\mathbb{E}_{\Pi}\Big[CY_{\omega} - A_{\omega}Q_{\omega} + \dfrac{3B}{4}Q^2_{\omega}\Big] \text{, subject to } \ Q_{\omega} = Y_{\omega} \text{ and } \ 0 \le Y_{\omega} \le x. \\ \max\limits_{\Pi} \ \mathbb{E}_{\Pi}\Big[CY_{\omega} - A_{\omega}Q_{\omega} + \dfrac{B}{2}Q^2_{\omega}\Big] \text{, subject to}\, \Pi \in \mathcal{D}_{GD}(\gamma). \end{array} \tag{49}$$

We computed this duopoly example using the path solver (REF). The total optimal investment of the producers is $x_1 + x_2 = 234$ MW. We report the solution of the complete-duopoly case in Table 9.

The financial market is complete and hence the risk probability measure is the identical for all agents (solution of the risk problem (44)). We report the expected net margin of the investment and the contribution of the strategic premiums. When the producer is risk-averse, the expected margin is not only coming from its strategic behavior but also from its risk aversion. The total welfare is equal to 345.4 M€: the surplus of the duopoly is 172.7 M€and the one of the consumer is 172.7 M€.

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	160	193	227	234	234	209.7
P [€/MWh]	140	156	173	216	266	190
Investment margin [€/kW]	388	604	867	1273	1711	969
- strategic margin [€/kW]	478	698	964	1025	1025	838
- risk margin [€/kW]	-90	-94	-97	248	686	131
Π	0.3406	0.2748	0.1967	0.1266	0.0613	0.2

Table 9: Equilibrium solution for the Cournot case (duopoly) with complete markets

4.5 Risky Cournot capacity equilibrium problem with incomplete markets

4.5.1 Illustrative example of Cournot equilibrium with incomplete markets

We elaborate the previous example, section 4.4.1, in the context of incomplete risk trading. This problem is not amenable to an optimization problem and was solved using PATH.

No risk trading When the three agents (2 producers + 1 consumer) can not trade risk, the total optimal investment of the producers drops to $x_1 + x_2 = 230$ MW. The equilibrium solution is reported in Table 10.

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	160	193.3	226.6	230	230	208
P [€/MWh]	140	156.6	173.3	220	270	192
Investment margin [€/kW]	398	621	888	1312	1750	993.6
- strategic margin [€/kW]	488	712	978	1007	1007	838
- risk margin [€/kW]	-90	-91	-90	305	743	155
$\Pi_1 = \Pi_2$ sym. producers	0.3228	0.2767	0.2217	0.1345	0.0443	0.2
Π_3 consumer	0.3690	0.2609	0.1325	0.1188	0.1188	0.2

Table 10: Solution of the Cournot equilibrium without risk trading

There is no risk trading: the producers and the consumer value the different scenario differently. The symmetric producers require a high expected margin for investing (993.6 €/kW): an important part is still due to their strategic behavior (838 €/kW) but the risk attitude plays a also a role. The welfare is equal to 341.7 M€, which is shared between the strategic duopoly (172.7 M€) and the consumer (169 M€).

One financial option We analyze the effect of imperfect trading by allowing the producers and the consumer to trade one financial option (cf., previous example). The total optimal investment of the duopoly becomes $x_1 + x_2 = 234.4$ MW. The solution (Table 11) is close

	scen 1	scen 2	scen 3	scen 4	scen 5	$\mathbb{E}_\Theta[\ ]$
Q [MWh]	160	193.3	226.6	234	234	209.6
P [€/MWh]	140	156.6	173.3	215.6	265.8	190.2
Investment margin [€/kW]	389	609	871	1273	1713	971.1
- strategic margin [€/kW]	479	700	962	1025	1025	838
- risk margin [€/kW]	-90	-91	-91	248	688	133
$\Pi_1 = \Pi_2$ sym. producers	0.3413	0.2725	0.1969	0.1306	0.0587	0.2
Π_3 consumer	0.3299	0.2953	0.1940	0.0904	0.0904	0.2

Table 11: Equilibrium solution of the Cournot case with one financial option

to the case where agents have access to a complete financial market (also in terms of net margin for investment). The total welfare is equal 345 M€, where the strategic duopoly risk-adjusted profit is 172.6 M€and the risk-adjusted surplus of the consumer is 172.4 M€.

4.5.2 Comparison of examples

We compare the different solutions in Table 12, where the cases are sorted by decreasing welfare. This allow us to highlight the effects of risk-aversion, Cournot competition and market incompleteness on the equilibrium.

In this simple setting, risk-aversion (calibrated on a Sharpe ratio of 0.5) has the biggest impact on welfare. Risk-aversion is not an economic inefficiency but a description of the utility function of the agents. Still its impact on investment is noticeable: comparing the risk neutral case to no risk trading, the drop in capacity is ≈ 12%. Complete trading partially alleviates this drop (by ≈ 2%); this relatively small welfare increase is due to the fact that the uncertain parameter in our simple example induces a similar risk exposure for the consumer and the producer, i.e., the high spot electricity prices occur in scenario where electricity has also a high utility for the consumer, while low electricity price in scenario where the utility is low. Hence, risk trading between the demand and the supply side cannot improve the situation by much. The impact of risk trading may be more sensitive to changes in other

Risk-attitude	Spot market	Financial market	Welfare [M€]	Capacity [MW]	Risk margin [€/kW]
Neutral	Competitive	-	490.9	389	0
Neutral	Cournot	-	436.1	259	0
Averse	Competitive	Complete	388.2	349	142
Averse	Competitive	1 option	387.7	348	144
Averse	Competitive	No risk trading	375.8	339	176
Averse	Cournot	Complete	345.4	234.4	131
Averse	Cournot	1 option	345.0	234	133
Averse	Cournot	No risk trading	341.7	230	155

Table 12: Welfare, capacity installed and risk margin

uncertainty parameters (e.g., the B parameter in the utility function or the technological cost), as shown in [8].

The strategic duopoly retains investment the biggest but the associated drop in welfare is less consequent : the loss of consumer surplus is compensated by the positive risk-adjusted profit of the duopoly (equal to 217 M€ in the risk neutral case and 172.7 M€ for all risk averse cases). Also in this case it appeared possible to restore the inefficiency of market incompleteness by introducing one financial option that hedges the negative demand shocks. Finally, Table 12 reports the investment margin due to risk, which we define as the expected margin of the investment minus the expected strategic margin (for the duopoly cases). This risk margin increases with market incompleteness, as the producer has difficulty in hedging the uncertain payoff of its investment. Also this risk margin is lower in the duopoly cases, the investment payoff being less risky due to strategic behavior.

5 Multistage risky competitive capacity equilibria

This section presents notation for risky design equilibria, and the specific situation of stochastic capacity investment equilibrium problems, when the uncertain outcomes occur in multiple stages and so do the investment decisions. The contribution of this section is in two parts. One aspect, though important, is almost entirely notational: We show that the analysis of the multistage case can be recovered directly from the two stage case. The case of multistage risky design is given by Theorem 5 while Theorem 6 applies this to multistage risky competitive capacity equilibria. The other contribution is a semi-realistic multistage numerical example of capacity equilibrium staged over a seven year period involving 9 million variables and 24 million nonzero elements in the associated matrices; see section 5.1.2.

The nearest precedents to this section are [11] and [41]. Both analyze multistage competitive situations in which risk aversion and risk trading are present at every stage. The former takes a social planners view of capacity investment while the latter gives an economic equilibrium interpretation to a social planner's multistage water planning problem for hydro power.

5.1 Multistage notation

5.1.1 Multistage notation relevant to expectations and CRMs

The phrase *in the multistage setting* refers to the following situation: Multistage uncertainty is represented by a (finite cardinality) tree of nodes or vertices

$$\omega \in \Omega^\circ := \Omega \cup \{\omega^\circ\},$$

i.e., a nonempty set of scenario nodes Ω augmented by a root scenario ω° together with a successor relation $\mathcal{S}(\cdot)$ which is a set mapping from Ω° to subsets of Ω including the empty set, where $\mathcal{S}(\omega)$ is the set of vertices that are immediate successors or children of node ω.[11] $\mathcal{L}$ denotes the set of leaves, i.e.,

$$\mathcal{L} := \{\omega \in \Omega \,|\, \mathcal{S}(\omega) = \varnothing\}.$$

Last, for any ω, define $[\omega]$ as the set of nodes on the predecessor path that starts at ω, moves to its unique immediate predecessor $(\mathcal{S})^{-1}(\omega)$[12] and continues recursively until it reaches ω°.

A *multistage uncertain outcome* or *multistage risky cost* is a vector $Z = (Z_\omega)_{\omega\in\Omega} \in \mathbb{R}^K$ where K is the cardinality of Ω. We also write L and L^c for the cardinalities of $\mathcal{L}$ and its complement $\mathcal{L}^c := \Omega^\circ \setminus \mathcal{L}$, so that $L^c = K + 1 - L$.

In addition we include a null stage quantity $Z_{\omega^\circ} := 0$ for the purpose of recursively calculating expectations and risk measures. A *multistage probability measure* $\Pi = (\Pi_\omega)_{\omega\in\Omega}$ is a dual vector whose action on $\mathcal{Z}$ is recursive (or conditional) as defined by a *multistage expectation*:

$$\mathbb{E}_\Pi[Z] := \mathbb{E}_\Pi[Z, \omega^\circ]$$

where for each $\omega \in \Omega^\circ$,

$$\mathbb{E}_\Pi[Z,\omega] := \begin{cases} Z_\omega & \text{if } \omega \text{ is a leaf,} \\ Z_\omega + \displaystyle\sum_{\omega' \in \mathcal{S}(\omega)} \Pi_{\omega'} \mathbb{E}_\Pi[Z,\omega'] & \text{otherwise.} \end{cases}$$

[11] We can relate this to stages by defining stage 0 to be the single node ω° and stage $j \ge 1$ to be the set of successors of all nodes in stage $j-1$.

[12] The tree structure of Ω° gives uniqueness of immediate predecessors.

By $\mathcal{P}$ we now mean the set of all multistage probability measures.

In Section 5.1.3 we will reformulate multistage probability measures as standard "single stage probability measures" on the space of uncertain outcomes over the event set $\mathcal{L}$, which is our route to using results in prior sections. By contrast, a multistage probability measure Π is not immediately recognizable as a probability measure on $\mathbb{R}^K$ even though it corresponds to a dual element $\nu \in \mathbb{R}^K_+$, where ν_ω is the product of $\Pi_{\omega'}$ over $\omega' \in [\omega]$, because the components of ν do not sum to 1 when there is more than one stage beyond the root node.[13]

We are interested in recursive formulations of multistage risk averse equilibrium problems, hence the following type of risk measure as developed in [49, Section 6.7.3].

Definition 5 (**Composite CRMs**)**.** *In the multistage setting, a* composite CRM r *is a real function on* $\mathcal{Z} = \mathbb{R}^K$ *if it has the following construction:*

1. *For each non-leaf node $\omega \in \Omega$, $\mathcal{D}(\omega)$ is a nonempty closed convex set of (single stage) probability measures on the event set $\mathcal{S}(\omega)$.*

2. *r is recursively defined for $\omega \in \Omega^\circ$ by $r(Z) := r(Z, \omega^\circ)$ where*

$$r(Z,\omega) \;:=\; \begin{cases} Z_\omega & \text{if } \omega \text{ is a leaf,} \\ Z_\omega + \sup\limits_{\Pi \in \mathcal{D}(\omega)} \sum\limits_{\omega' \in \mathcal{S}(\omega)} \Pi_{\omega'} r(Z,\omega') & \text{otherwise.} \end{cases} \tag{50}$$

Note that a composite CRM is the support function of a nonempty closed convex set, $\mathcal{D}$, of multistage probability measures Π such that, for each $\omega \in \mathcal{L}^c$, $\Pi_{\mathcal{S}(\omega)} := (\Pi_{\omega'})_{\omega' \in \mathcal{S}(\omega)}$ lies $\mathcal{D}(\omega)$. In other words, the *multistage or composite risk set* $\mathcal{D}$ of a composite CRM is the stagewise Cartesian product of (single stage) risk sets. Conversely, if $\mathcal{D}$ is a nonempty closed convex set of multistage probability measures then its composite action described above follows by by setting $\mathcal{D}(\omega) := \{(\Pi_{\mathcal{S}(\omega)} : \Pi \in \mathcal{D}\}$ for each $\omega \in \mathcal{L}^c$. A trivial example is that a multistage probability measure Π gives a composite CRM r by defining $\mathcal{D}(\omega)$ as the singleton $\{\Pi_{\mathcal{S}(\omega)}\}$ for each $\omega \in \Omega^\circ \setminus \mathcal{L}$, hence $r = \sigma_{\mathcal{D}} = \mathbb{E}_\Pi$.

Extending the minimization representation (1) of a CRM, it may be convenient to work with composite CRMs that are computed by nested minimization. Similar to (50), we are interested in the case when $r(Z)$ can be computed as $\hat{r}(Z,\omega^\circ)$ where $\hat{r}(Z,\omega) := Z_\omega$ for $\omega \in \mathcal{L}$ while for $\omega \in \mathcal{L}^c$,

$$\begin{aligned} &\hat{r}(Z,\omega) \\ &\;:=\; \min\Big\{ \quad g_\omega\big(u_\omega, \hat{Z}(w)\big) \\ &\qquad\qquad \text{subject to} \quad \hat{Z}(\omega) = \big(\hat{r}(Z,\omega')\big)_{\omega' \in \mathcal{S}(\omega)},\; G\big(u_\omega, \hat{Z}(\omega)\big) \in \mathcal{K}_\omega \quad \Big\} \end{aligned} \tag{51}$$

[13] E.g., in a binary tree, $\sum_{\omega \in \Omega} v_\omega$ is the number of stages beyond the root node.

where g_ω and G_ω are smooth finite dimensional mappings and $\mathcal{K}_\omega$ is a finite dimensional nonempty closed convex cone. This allows for a composite CRM to be constructed by nested evaluation of standard CVaR or good deal CRMs.

Example 3. Multistage CV@R (cf., Example 1 in section 2.) Here the multistage risk set defined at each stage ω by

$$\mathcal{D}_{\text{CV@R}}(\omega) \;:=\; \{\Pi(\omega) \text{ is a probability measure} | \Pi_{\omega'} \le \gamma(\omega)^{-1}\Theta_{\omega'} \text{ for all } \omega' \in \mathcal{S}(\omega)\}.$$

The convex minimization formulation of the multistage CV@R is the following LP.

$$\begin{aligned}\hat{r}_{\text{CV@R}}(Z,\omega) := \min_{t(\omega),U(\omega)} \Big\{ \quad & t(\omega) + \gamma(\omega)^{-1}\mathbb{E}_{\Theta(\omega)}[U(\omega)] \\ \text{subject to} \quad & U(\omega) \in \mathbb{R}_+^{|\mathcal{S}(\omega)|}, \\ & U(\omega) - \hat{Z}(\omega) + t(\omega) \in \mathbb{R}_+^{|\mathcal{S}(\omega)|}, \\ & \hat{Z}(\omega) = \left(\hat{r}_{\text{CV@R}}(Z,\omega')\right)_{\omega' \in \mathcal{S}(\omega)} \quad \Big\}\end{aligned} \tag{52}$$

Example 4. The multistage good deal, [11] (cf., Example 2 in section 2) has a multistage risk set defined at stage ω by

$$\mathcal{D}_{GD}(\omega) := \left\{ \Pi(\omega) \text{ is a probability measure} \;\middle|\; \mathbb{E}_{P\omega)}\left[\left(\frac{\Pi(\omega)}{\Theta(\omega)}\right)^2\right] \le \gamma(\omega)^2 \right\}$$

The convex minimization formulation of the multistage good-deal is the following SOCP (cf., (3) in section 2),

$$\begin{aligned}\hat{r}_{GD}(Z,\omega) := \min_{s(\omega),t(\omega),U(\omega)} \Big\{ \quad & t(\omega) + \gamma(\omega)^{-1}s(\omega) \\ \text{subject to} \quad & U(\omega) \in \mathbb{R}_+^{|\mathcal{S}(\omega)|}, \\ & U(\omega) - \hat{Z}(\omega) + t(\omega) \in \mathbb{R}_+^{|\mathcal{S}(\omega)|}, \\ & \left(\sqrt{\Theta(\omega)}U(\omega), s(\omega)\right) \in \mathbb{L}^{|\mathcal{S}(\omega)|+1}, \\ & \hat{Z}(\omega) = \left(\hat{r}_{GD}(Z,\omega')\right)_{\omega' \in \mathcal{S}(\omega)} \quad \Big\}\end{aligned} \tag{53}$$

This problem has L_c second-order cone constraints.

5.1.2 Fishbone example - computational experiment

The models presented permit to analyze how risk affects the market equilibrium. They are specially relevant for energy markets, as capacities are long-lived assets particularly vulnerable to long-term risk. As long as they are amenable to convex optimization formulations (e.g., linear program for a polyhedral risk measure like CV@R or SOCP for the good-deal), they can include all the relevant details of the market microstructure (as traditional capacity expansion model).

We provide here a numerical example inspired by the situation of European countries. The electricity demand faced a negative shock starting 2009 and has not yet recovered pre-crisis level. Electricity producers facing such low demand and depreciated prices, reacted by mothballing or decommissioning some of their plants (about 24 GW is mothballed and 7 GW is retired in Europe). As soon the economy (and the electricity demand) would recover, the de-mothballing of those assets should also occur. The timing of those actions is therefore affected by the risk of the macro-economic situation. We model this situation by constructing the following "fishbone" tree where each year there is a probability of 50% of staying in a central scenario fed with our best macroeconomic estimates (gradual rebalancing of the electricity demand, fuel and CO_2 prices,...), a probability of 25% to move to a more optimistic situation and a probability of 25% to go to a more pessimistic situation (cf. Figure 1). This scenario tree reduces the numerical complexity of the problem (and avoids the curse of dimensionality) while retaining the dynamic effects of uncertainty in the backbone scenario.

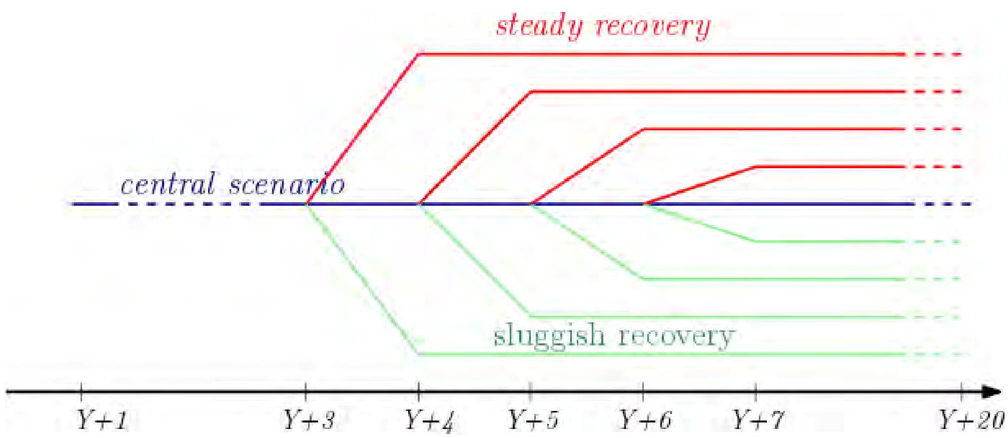

Figure 1: The scenario tree

The experiment is designed to analyze how risk affects a competitive electricity market and how trading opportunities mitigate the effects. We compute the two counterfactual capacity equilibria that span the extremes of risk trading, namely with a complete financial market (where every risk can be traded between agents) and with an incomplete market in which there are no financial contracts. These are both cases of risky competitive capacity equilibrium problems in Definition 3, section 3, albeit for a multistage rather than a two stage setting.

To further simplify the experiment we assume that there are only two representative agents acting on the electricity market: a price taking representative producer, owning and managing all the capacities and a representative consumer, with a linear utility function $U(Q) = VOLL \times Q$. This consumer only consumes nonnegative quantity Q below a reference load. This consumer model leads to the common inelastic demand up to $VOLL$. The agents risk aversion is modeled by the good-deal risk measure calibrated on the average Sharpe ratio of S&P 500 index[14]. The reference load is described by a load duration curve with 144 time segment per year. At the beginning of the horizon, the producer owns about 210 assets that can be actively managed (mothballing, early retirement, technological reconversion). All those short-term decisions have a lead time of a year. The producer can also invest in new assets (relevant for the second decade). To sum up, the models have roughly 9 million variables. The complete market is solved numerically by the equivalent SOCP program (about 24 million nonzero elements in the matrix).

The incomplete market is solved by an iterative scheme that is a kind of alternating method. Each iteration first solves the risk neutral problem given the probability measure Π of the producer, and then updates this probability measure based on the outcome of the risk neutral problem. This alternating method uses the fact that the risk aversion of the consumer does not impact the equilibrium (the consumer does not have time-dependent decision) and there is only one price taking producer so that, given Π, one can compute the equilibrium by solving the equivalent risk neutral optimization problem as a very large LP. This computational heuristic converged well in our settings.

Figure 2 depicts the optimal asset management for the producer in the central scenario (i.e., the backbone of the fishbone tree). It shows how complete risk trading between the consumer and producer incentivize the producer to de-mothball its gas assets more quickly than if he would not have the possibility to trade risk. This is consistent with accepted economic theory [39] that a market satisfying all assumptions of perfect competition in a deterministic world may become grossly inefficient in a risky environment if proper instruments for trading risk do not exist. A final note is that what may seem a relatively small difference, of 5€/MWh, in the baseload price between the complete and incomplete cases is significant when considering the profit margin rather than revenue of any generating firm.

The inefficiency of the incomplete case is even appearing when analyzing the baseload price in Figure 3. Even if it is higher than in the complete market case it still does not incentivize well the producer to de-mothball its gas assets.

5.1.3 Transformation from multistage setting to single stage setting

To prepare for using results from previous sections, we translate of $\mathcal{Z}$ to a space of "single stage uncertainties", $\mathcal{Z}(\mathcal{L}) := \mathbb{R}^L$, where L is the number of leaf nodes. Allowing for different

[14] This Sharpe ratio was equal to 0.53 at the time of computation

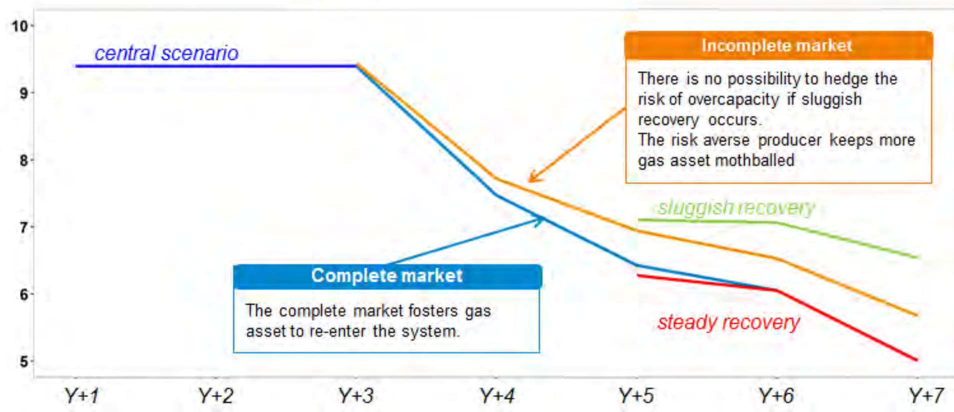

Figure 2: Mothballed capacity of gas assets [GW]

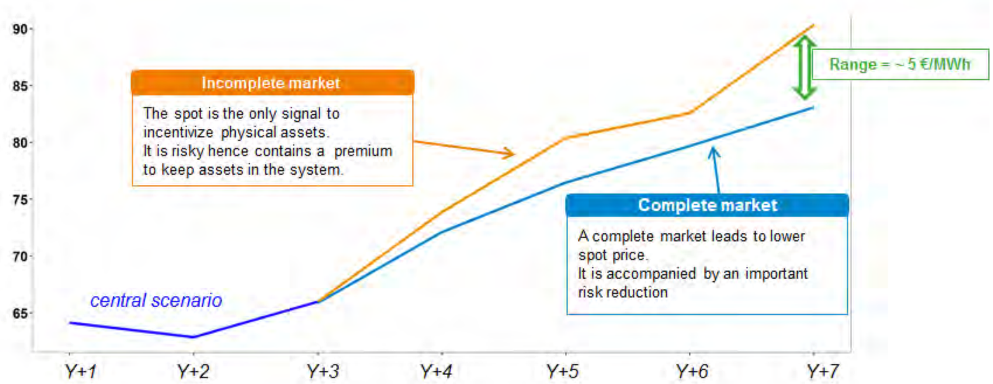

Figure 3: Baseload price [EUR/MWh]

notation, what follows and indeed further details can be found in [49, Section 6.7.3].

The translation of a multistage risky cost $Z \in \mathcal{Z}$ to a "single stage risky cost" $Z^{\mathcal{L}}$ is via sums over predecessor paths,

$$Z_\omega^{\mathcal{L}} := \sum_{\omega' \in [\omega]} Z_{\omega'} \quad \text{for } \omega \in \mathcal{L} \tag{54}$$

where $Z_{\omega^\circ} := 0$ as previously. The translation of a multistage probability measure Π, on $\mathcal{Z}$, to a probability measure $\Pi^{\mathcal{L}}$, on $\mathcal{Z}(\mathcal{L})$, where we call the latter a single stage probability measure, is via products over predecessor paths,

$$\Pi_\omega^{\mathcal{L}} := \prod_{\omega' \in [\omega]} \Pi_{\omega'} \quad \text{for } \omega \in \mathcal{L}$$

which relies on the auxiliary quantity $\Pi_{\omega^\circ} := 1$.

Two easy checks confirm that this translation is sensible: If Π is a multistage probability measure then, first, the components of $\Pi^{\mathcal{L}}$ are nonnegative and sum to 1; and second, the multistage action (below left) is consistent with the single stage action (below right):

$$\mathbb{E}_{\Pi}[Z] = \sum_{\omega \in \mathcal{L}} \Pi^{\mathcal{L}}_{\omega} Z^{\mathcal{L}}_{\omega}.$$

We also need notation translating a multistage risk set $\mathcal{D}$ to its single stage counterpart,

$$\mathcal{D}^{\mathcal{L}} := \{\Pi^{\mathcal{L}} : \Pi \in \mathcal{D}\}.$$

Here we give a summary of properties for multistage risk sets and CRMs that will need later, details of which can be found in [49, Section 6.7.3].

Lemma 2. *In the multistage setting:*

1. *If $\mathcal{D}$ is a closed convex set of multistage probability measures then $\mathcal{D}^{\mathcal{L}}$ is a closed convex set of (single stage) probability measures.*

2. *If $\mathcal{D}_i$ is a closed convex set of multistage probability measures for $i \in \mathcal{I}$, then*

$$\Big(\bigcap_{i \in \mathcal{I}} \mathcal{D}_i\Big)^{\mathcal{L}} = \bigcap_{i \in \mathcal{I}} \mathcal{D}^{\mathcal{L}}_i$$

3. *r is a composite CRM on $\mathcal{Z}$ if and only if r is the support function of a multistage risk set $\mathcal{D}$ that is closed, convex and nonempty. Indeed, such a composite CRM $r = \sigma_{\mathcal{D}}$ is equivalent to CRM $r^{\mathcal{L}} = \sigma_{\mathcal{D}^{\mathcal{L}}}$ via the identity*

$$\sigma_{\mathcal{D}}(Z) = \sigma_{\mathcal{D}^{\mathcal{L}}}\big(Z^{\mathcal{L}}\big) \ \textit{for all } Z \in \mathcal{Z}.$$

5.2 Multistage risky design problems

To define a multistage decision we need multistage variables which are vectors of stagewise design variables, one for every non-leaf node. Recall the set of leaf nodes $\mathcal{L}$, with cardinality L, and its complement $\mathcal{L}^c$ in Ω°, with cardinality L^c. Formally, a multistage decision x is a vector consisting of one subdecision, a subvector x_ω, for each non-leaf node:

$$x := \big(x_\omega\big)_{\mathcal{L}^c} := \big(x_\omega\big)_{\omega \in \mathcal{L}^c}.$$

There is no loss of generality in assuming x_ω is an n-dimensional vector for each ω; this simplicity is directly applicable in the setting of capacity decisions where the same technologies are available at each node. That is,

$$x \in \mathbb{R}^{nL^c}.$$

For instance in the two stage setting we have $\mathcal{L}^c = \{\omega^\circ\}$, i.e., the only investment is x_{ω° at stage 0 (as in prior sections).

We will consider various agents i each with a multistage decision $x_i := \left(x_{i\omega}\right)_{\mathcal{L}^c}$. Then the multistage (complete) risk market has the same form as given in section 2: (4) (for all i) and (5). Likewise, the multistage risky Nash equilibrium problem has the same general format as previously: (6) (for all i) with (8).

To interpret the design problem faced by agent i as a multistage decision problem, its risky cost mapping $\Xi_i(x_i, x_{-i})$ must respect the staged nature of the uncertainty, namely, it is nonanticipative as described later. While nonanticipativity is needed for a practical or useful formulation of an agent's multistage optimization problem, neither that restriction nor the lack of it affects the mathematical relationships between various equilibrium reformulations explored in previous sections.

We now catalog the structural conditions required for the multistage design equilibrium problems under risk aversion, where the Design Convexity and Design Technical Assumptions are tacit.

Multistage assumptions. We are in the multistage setting with notation given in section 5.1.1.

1. Each X_i is a set of vectors $x_i := \left(x_{i\omega}\right)_{\omega \in \mathcal{L}^c}$.
2. The CRM assumptions of section 2 hold with "composite CRM" and "multistage risk set" replacing "CRM" and "risk set" throughout.

In light of Lemma 2, Theorem 1 from section 2 can be directly applied to multistage risky design equilibrium problems as we do next in Theorem 5. In fact we don't need Theorem 5 in the sequel. We provide it because it is a template for everything that follows, namely, single stage results generally yield multistage counterparts without further work. See the Appendix for a proof.

Theorem 5. *Under the Design Convexity, Design Technical and Multistage (including CRM) Assumptions:*

1. *The multistage risky design equilibrium problem and the multistage risky Nash game are equivalent:* $(x_{\mathcal{I}}, P^{\mathrm{r}})$ *is a multistage risky design equilibrium if and only if, for* $\Pi = P^{\mathrm{r}}$, $(x_{\mathcal{I}}, \Pi)$ *solves the multistage risky Nash game.*

2. *An equilibrium exists for both the multistage risky design equilibrium problem and multistage risky Nash game.*

To complete the formalities we describe nonanticipative optimization, namely, a problem structure that respects the order in which information arrives during the decision making process. Since Ω_0 is a tree, each (non-root) $\omega \in \Omega$ has a unique immediate predecessor $\omega' = (\mathcal{S})^{-1}(\omega)$. Thus each node $\omega \in \Omega$ has is a (unique and well defined) predecessor path that begins at ω and runs back to the root ω°, which we denote by $[\omega]$. A mapping Ξ from x to $\mathcal{Z}$ that respects the stochastic tree structure must have the following form for each component $\omega \in \Omega$:

$$\Xi_\omega(x) \;:=\; \Xi_\omega\big(x^p(\omega)\big) \tag{55}$$

where

$$x^p(\omega) \;:=\; \big(x_{\omega'}\big)_{\omega' \in [\omega]}. \tag{56}$$

This enforces nonanticipativity on $\Xi_\omega(x)$ in that its value depends only on x_ω and other decisions on its predecessor path.

In practice, we would apply Theorem 5 to risky design equilibrium problems for which each agent optimizes a nonanticipative design problem.

5.3 Multistage risky competitive capacity equilibria with complete markets

Here we extend the two stage competitive capacity setting with complete markets of section 3.1.2 to multiple stages. Recall the two perfectly competitive agents, a genco and a retailer of electricity, hence $\mathcal{I} = \{1, 2\}$. Given a multistage capacity decision $x_i := (x_{i\omega})_{\mathcal{L}^c}$, agent i's investment cost is $I_i(x_i)$. (In simplest form, $I_i(x_i) \;:=\; \sum_{\omega \in \mathcal{L}^c} I_{i\omega}(x_{i\omega})$.) In this multistage setting, the capacity of any agent i at node ω is the sum of the prior investments on its predecessor path:

$$\xi_{1\omega} \;=\; \sum_{\omega' \in [\omega] \setminus \omega} x_{1\omega'}, \tag{57}$$

$$\xi_{2\omega} \;=\; \sum_{\omega' \in [\omega] \setminus \omega} x_{2\omega'}. \tag{58}$$

A simple instance of multistage constraints on both investments and stochastic scenario recourse decisions of the generator would be to combine (57) with recourse constraints such as $0 \le Y_{i\omega} \le \xi_{i\omega}$ for each $\omega \in \Omega$. Such multistage constraints are an example of (9) for suitable $A_{1\omega}$, $B_{1\omega}$ and $b_{1\omega}$. Thus the Competitive Spot Market Assumptions of section 3.1, though posed in the two stage setting, remain appropriate to the multistage setting. A further note is that while nonnegativity would be natural for physical capacity, via $\xi_i \ge 0$, negative components in x_i could be allowed to model the retirement or mothballing of capacity at any stage.

At node $\omega \in \Omega$, given capacity $\xi_{1\omega}$ and commodity price P_ω, the generator solves the recourse problem $V_{1\omega}(\xi_{1\omega}, P_\omega)$ given by (11) but with $\xi_{1\omega}$ replacing x_0. Likewise at node ω, the retailer, given capacity $\xi_{2\omega}$ and commodity price P_ω, has the recourse problem $V_{2\omega}(\xi_{2\omega}, P_\omega)$ defined by (12); and the social planner, given capacity $(\xi_{1\omega}, \xi_{2\omega})$, has the recourse problem $V_{0\omega}(\xi_{1\omega}, \xi_{2\omega})$ given by (14).

The multistage version of the generator's and retailer's problems, (16), is

$$\begin{array}{rl} \min\limits_{x_1, Y, W_1} & I_1(x_1) + P^{\mathbf{r}}[W_1] + r_1\big(C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega - W_{1\omega}\big) \\ \text{subject to} & x_1 \in X_1, (9) \text{ and } (57) \text{ hold for all } \omega, \\ \min\limits_{x_2, Q, W_2} & I_2(x_2) + P^{\mathbf{r}}[W_2] + r_2\big(P_\omega Q_\omega - U_\omega(Q_\omega) - W_{2\omega}\big) \\ \text{subject to} & x_2 \in X_2, (10) \text{ and } (58) \text{ hold for all } \omega. \end{array} \tag{59}$$

The "design variables" comprise (x_i, ξ_i) and X_i could be any nonempty closed convex set in $\mathbb{R}^{L^c}$t.The multistage stochastic equilibrium price P has the same specification as previously, i.e., for each spot market scenario $\omega \in \Omega$, P_ω must be an equilibrium spot price corresponding to the quantities Y_ω, Q_ω. As before, the equilibrium condition on $P^{\mathbf{r}}$ is implicit in the balance of risk trades: $\sum_i W_i = 0$.

Moreover each composite CRM can be evaluated via nested minimization (51) rather than nested maximization (50). This would put (59) in the form of a traditional multistage stochastic program.

The next result is actually a corollary of Theorem 3, i.e., the equivalence between risky competitive capacity equilibrium and risk averse optimization problems is valid in the multistage setting. A proof is given in the Appendix.

Theorem 6 (**Multistage risky competitive capacity equilibrium with complete markets as multistage risk averse optimization).**
If the multistage CRM Assumptions with $\mathcal{I} := \{1, 2\}$ and the Competitive Spot Market Assumptions hold then (x_1, ξ_1, x_2, ξ_2), with some $(W_1, W_2, P, P^{\mathbf{r}})$, is a multistage complete competitive capacity equilibrium if and only if (x_1, ξ_1, x_2, ξ_2) solves the risky system capacity problem

$$\begin{array}{rl} \min\limits_{x_1, \xi_1, Q, x_2, \xi_2, Y} & I_1(x_1) + I_2(x_2) + r_0\big(C_\omega(Y_\omega) - U_\omega(Q_\omega)\big) \\ \text{subject to} & x_1 \in X_1, x_2 \in X_2, (9), (10), (57), (58) \textit{ hold for all } \omega. \end{array} \tag{60}$$

6 Existence of risky capacity equilibria with incomplete markets

In this section we give sufficient conditions for existence of capacity planning equilibria with incomplete markets for both competitive spot markets and Cournot spot markets (recall

Definitions 3 and 4 in sections 3 and 4 respectively). We pose incompleteness by confining risk trades to lie in a proper subspace $\mathcal{W}$ of $\mathcal{Z}$. This kind of incompleteness occurs naturally in financial markets where the various classes of traded financial securities effectively provide a basis of the vector space of all possible trades, $\mathcal{W}$. In energy markets, members of $\mathcal{W}$ include forward contracts on fuel prices for thermal generators, forward contracts on electricity prices. However traded products typically aren't available for very long term contracts [18] and other important uncertainties such as regulatory shifts, e.g., the existence or level of prices on CO_2 emissions in North America, or prices of European emissions permits beyond more than a year.

The literature on how risk aversion and financial market incompleteness impacts capacity expansion markets is small. We highlight a very recent paper [1] which shows existence of competitive capacity equilibria problems with incomplete markets via degree theory. We give existence of such equilibria in Theorem 8. Our proof is considerably simpler that of [1] because we draw on prior work on incomplete design markets [43]. This allows an immediate and standard application of Kakutani's fixed point theorem — which goes back to Nash's seminal approach [36] — to prove existence.

Subspace incompleteness, via $\mathcal{W}$ as above, is just one form of financial market incompleteness. A relatively early paper [12] explores various equilibrium models for capacity expansion accounting for different risk aversion characteristics of agents but no risk trading. Liquidity limits, represented by a bound on elements of $\mathcal{Z}$, is another possibility, see [9]. [8] combines subspace incompleteness with liquidity limits.

In section 6.1 we review risky design equilibria with incomplete markets based on [43, Section 4] in order to introduce *marginal risk equilibria*, see Definition 6. Marginal risk equilibria are important in reducing equilibrium models with incomplete risk trading to a format that is amenable to application of classical existence results.

Section 6.2 tackles the competitive equilibrium problem by introducing its *marginal* form in Definition 7 and using it to show existence of solutions in Theorem 8. For the Cournot case see section 6.3 where the marginal form appears in Definition 8 and existence of solutions in Theorem 9 as a consequence of Nash [36].

6.1 Review of risky design problems with incomplete markets

Here we present the findings of [43, Section 4] in which marginal risk equilibria are used to characterize the incomplete version of risky design equilibria. This is an extension of the review of risky design equilibria with complete markets and their relationship to complete design games in section 2.

We are given stochastic processes $\Xi_i(x_i, x_{-i})$, representing uncertain cost, for each agent

$i \in \mathcal{I}$. We reprise (4): Agent i has a decision problem that combines design ($x_i \in \mathbb{R}^n$) and incomplete hedging ($W_i \in \mathcal{W}$),

$$\min_{x_i, W_i} P^{\mathrm{r}}[W_i] + r_i\big(\Xi_i(x_i, x_{-i}) - W_i\big) \quad \text{subject to} \quad x_i \in X_i,\ W_i \in \mathcal{W}. \tag{61}$$

Recall from Definition 1.3 that if $x_{\mathcal{I}} \in \mathbb{R}^{nN}$, $W_{\mathcal{I}} \in \mathcal{W}^N$, $P^{\mathrm{r}} \in \mathcal{W}^*$ are such that each (x_i, W_i) solves (61) and $W_{\mathcal{I}}$ is balanced, then $(x_{\mathcal{I}}, W_{\mathcal{I}}, P^{\mathrm{r}})$ is an *risky design equilibrium with incomplete markets*.

The most extreme case of incompleteness is no risk trading, i.e., $\mathcal{W} = \{0\}$, this reduces to the *RA design game*,

$$\min_{x_i} r_i\big(\Xi_i(x_i, x_{-i})\big) \quad \text{subject to} \quad x_i \in X_i. \tag{62}$$

Existence of equilibria this risk averse case can be tackled by the wealth of existing results starting with Nash's seminal work [36] with further developments summarized in [14].

Next we introduce a risk neutral version of (61). We call it an *incomplete risk neutral (Nash) game*; it extends the standard risk neutral game (6) by allowing each agent to have a different probability measure Π_i:

$$\min_{x_i} \mathbb{E}_{\Pi_i}\big[\Xi_i(x_i, x_{-i})\big] \quad \text{subject to} \quad x_i \in X_i. \tag{63}$$

What links (61) and (63) is the notion of *marginal risk equilibrium*, defined next.

Definition 6 (Marginal risky equilibrium). *[43, Definition 7] Assume we are given subspace $\mathcal{W}$ of $\mathcal{Z}$ and risky costs $Z_{\mathcal{I}} \in \mathcal{Z}^N$. We say $\Pi_{\mathcal{I}}$ is a* marginal risk equilibrium *for $Z_{\mathcal{I}}$ (relative to $\mathcal{W}$) if there exist $P^{\mathrm{r}} \in \mathcal{W}^*$ and a balanced list of trades $W_{\mathcal{I}} \in \mathcal{W}^N$ such that for all i,*

$$\begin{array}{rcl} \Pi_i & \in & \partial r_i\big(Z_i - W_i\big) \\ P^{\mathrm{r}} & = & \Pi_i|_{\mathcal{W}}. \end{array} \tag{64}$$

The close relationship between (61) and (63) is shown next.

Theorem 7. *[43, Theorem 11] Suppose the Design and CRM assumptions of section 2 hold, and $\mathcal{W}$ is a subspace of $\mathcal{Z}$. If $\mathcal{D}_0$ has nonempty interior relative to $\mathcal{P}$ then $x_{\mathcal{I}}$ is, together with some $W_{\mathcal{I}}$ and P^{r}, a risky design equilibrium with incomplete markets if and only if*

1. *$x_{\mathcal{I}}$ solves the incomplete risk neutral design game* (63)*; and*

2. *$\Pi_{\mathcal{I}}$ is a marginal risk equilibrium for $Z_{\mathcal{I}} = (\Xi_i(x_{\mathcal{I}}))_{i \in \mathcal{I}}$.*

The system given by statements 1 and 2 makes no mention of the list of equilibrium risk trades $W_{\mathcal{I}}$; these are implicit in the requirement that $\Pi_{\mathcal{I}}$ is a marginal risk equilibrium. While this reformulation offers no advantage for computation, which still needs to specify risk trades, it is helpful in deriving existence of equilibria via Kakutani's fixed point theorem. We will use this result to reformulate the risk capacity equilibria with incomplete markets in both the competitive and Cournot spot market cases, and give mild sufficient conditions for existence of equilibria in each case.

We present two ancillary results from the proof of [43, Theorem 11]. The first explains that the above interiority condition on the system risk set is sufficient for the incomplete risk market to achieve equilibrium, indeed for boundedness of equilibrium risk trades. Although we don't make direct use of that result, it is implicit in everything that follows, e.g., Kakutani's fixed point theorem relies on boundedness of the relevant set-valued mapping.

Proposition 2. *[43, Theorem 11] Given an uncertain cost $Z_i \in \mathcal{Z}$ and a CRM r_i on $\mathcal{Z}$ for each $i \in \mathcal{I}$, if the system risk set $\mathcal{D}_0$ has nonempty interior relative to $\mathcal{P}$ then the set of incomplete risk market equilibria is nonempty and compact.*

The second result is for later use in establishing that the fixed point mappings used to represent incomplete capacity equilibria are closed.

Lemma 3. *[43, Theorem 11] In the situation of Theorem 7, the set mapping Φ_{r} which sends $Z_{\mathcal{I}} \in \mathcal{Z}^N$ to the set of all its marginal risk equilibria $\Pi_{\mathcal{I}}$ takes nonempty closed convex set values, and has a closed graph: If $Z^{\nu}_{\mathcal{I}} \to \hat{Z}_{\mathcal{I}}$ in $\mathcal{Z}^N$ and $\Pi^{\nu}_{\mathcal{I}}$ is a marginal risk equilibrium for $Z^{\nu}_{\mathcal{I}}$, for each ν, then each limit point of the sequence $\{\Pi^{\nu}_{\mathcal{I}}\}$ is a marginal risk equilibrium for $\hat{Z}_{\mathcal{I}}$.*

6.2 Existence of risky capacity equilibria with incomplete markets

We recall the formulation of the risky competitive capacity equilibrium problem with incomplete markets from section 3.1. Given capacity x_1 and x_2 of the generator and retailer, in spot scenario ω, the equilibrium quantities Y_ω and Q_ω are given by (11) and (12), and the corresponding equilibrium price P_ω is set by (13). In the risky competitive capacity equilibrium with incomplete markets (Definition 3.3), the agents determine (x_1, Y_1, W_1) and (x_2, Q_2, W_2) by solving (18), which constrains W_i to lie in $\mathcal{W}$; P is a stochastic spot equilibrium price; and P^{r} clears the risk market, i.e., $W_1 + W_2 = 0$ as given in (17).

Recall further the capacity and quantity decisions of producers in the risk neutral case of competitive capacity equilibrium, (15). The incomplete risk neutral formulation, by analogy to (63), is

$$\begin{aligned} &\min_{x_1,Y} \quad I_1(x_1) + \mathbb{E}_{\Pi_1}\big[C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega\big] \quad \text{subject to } \; x_1 \in X_1, (9) \text{ for all } \omega, \\ &\min_{x_2,Q} \quad I_2(x_2) + \mathbb{E}_{\Pi_2}\big[P_\omega Q_\omega - U_\omega(Q_\omega)\big] \quad \text{subject to } \; x_2 \in X_2, (10) \text{ for all } \omega. \end{aligned} \tag{65}$$

This is identical to (15) other than that, here, each probability measure Π_i may depend on the agent. It appears in the marginal version of competitive capacity equilibrium problem:

Definition 7 (**Marginal risky competitive capacity equilibrium problem).**
In the (two stage) marginal competitive capacity equilibrium problem *we seek*

(a) an investment x_i *and stochastic quantity* Y *or* Q *that, for* $i = 1, 2$, *solve* (65).

(b) Y_ω, Q_ω, P_ω *that solve* (11), (12), (13) *respectively, for all* ω*; and*

(c) (Π_1, Π_2) *that is a marginal risk equilibrium for* (Z_1, Z_2) *when* $Z_1 = (C_\omega(Y_\omega) - P_\omega\, e^\top Y_\omega)$ *and* $Z_2 = (P_\omega Q_\omega - U_\omega(Q_\omega))$.

In the situation of Theorem 7, it is immediate that the marginal risky competitive equilibrium problem is obtained from, indeed equivalent to, the risky competitive capacity equilibrium problem with incomplete markets. We state this formally in part 1 of the next theorem. Part 2 is our main goal, giving existence of risky competitive equilibria with incomplete markets under the next sufficient condition. This is an alternative to [1] which shows existence of equilibria in incomplete markets by applying degree theory to the stationary conditions that characterize the equilibrium problem. Our proof, which appears in the Appendix, is direct because posing the equilibrium problem in terms of marginal risk allows direct application of Kakutani's fixed point theorem. This elementary approach dates back to Nash's seminal paper [36].

Competitive Cost Growth Assumption

1. When $x_i \in X_i$, $I_i(x_i) \to \infty$ as $\|x_i\| \to \infty$.
2. For each ω, when $x_1 \in X_1$ and Y_ω that is feasible for (9),
   (a) $I_1(x_1) + C_\omega(Y_\omega) - U'_\omega(0;1)e^\top Y_\omega \to \infty$ as $\|(x_1, Y_\omega)\|_\infty \to \infty$, and
   (b) $\|x_1\| \to \infty$ as $\|Y_\omega\|_\infty \to \infty$.
3. For each ω, when $x_2 \in X_2$, $\omega \in \Omega$ and Q_ω is feasible for (10),
   (a) $I_2(x_2) - U_\omega(Q_\omega) \to \infty$ as $\|(x_i, Q_\omega)\|_\infty \to \infty$, and
   (b) $\|x_2\| \to \infty$ as $\|Q_\omega\|_\infty \to \infty$.

Similar to the Cournot Cost Growth Assumptions from section 4.4, this assumption reflects the practicalities of capacity markets. For example, part 2(a) says that combined capacity and production cost will, at high quantities, outstrip the revenue of production even at the highest price;[15] and 2(b), that increasing production implies increasing capacity.

Theorem 8. *If the CRM and Competitive Spot Market Assumptions hold, $\mathcal{W}$ is a proper subspace of $\mathcal{Z}$, and $\mathcal{D}_0$ has nonempty interior relative to $\mathcal{P}$, then*

1. (x_1, Y, x_2, Q) *and P, together with some (W_1, W_2) and P^{r}, form a risky competitive capacity equilibrium with incomplete markets if and only if* (x_1, Y, x_2, Q) *and P, together with some* (Π_1, Π_2)*, solve the marginal form in Definition 7 above; and*

2. *there exists a risky competitive capacity equilibrium with incomplete markets if either the Competitive Cost Growth Assumption holds or the feasible sets of* (65) *are bounded.*

See section 3.5 for a numerical example that illustrates two incomplete cases of competitive capacity equilibrium models: no risk trading ($\mathcal{W} = \{0\}$) and trading of a single kind of contract ($\mathcal{W}$ has dimension 1).

6.3 Existence of risky Cournot capacity equilibria with incomplete markets

We recall the setting of stochastic Cournot capacity equilibria from section 4.2, in which there are $N-1$ producers indexed by $i \in \mathcal{I} = \{1, \ldots, N-1\}$, and an Nth agent representing consumer demand. Consider the Cournot spot market in scenario ω. Producer i, with plant capacity x_i, optimizes production by choosing $y_i = Y_{i\omega}$ to solve (34). Given producers' decisions $Y_{\mathcal{I}\omega} = (Y_{i\omega})_{i\in\mathcal{I}}$, the consumer sets a price by looking at the total output and obtains a surplus $V^{\mathrm{C}}_{N\omega}(Y_{\mathcal{I}\omega})$ as in (35).

Recall further the capacity and quantity decisions of producers in the risk neutral Cournot capacity problem, (36). The incomplete version is similar but with a possibly different probability measure Π_i for each producer:

$$\begin{array}{ll} \min\limits_{x_i} & I_i(x_i) + \mathbb{E}_{\Pi_i}\big[R_{i\omega}(Y_{i\omega}, Y_{-i\omega})\big] \\ \text{subject to} & x_i \in X_i,\ 0 \le Y_{i\omega} \le x_i \text{ for all } \omega\,, \end{array} \tag{66}$$

Definition 8 (**Marginal risky Cournot capacity equilibrium problem).** *In the* (two stage) marginal Cournot capacity equilibrium problem *we seek (a) an investment x_i and stochastic quantity Y_i that, for each $i \in \mathcal{I}$, solve* (66) *such that (b) the quantity $Y_{i\omega}$ solves* (34) *for each $i \in \mathcal{I}$ and ω; and (c)* $(\Pi_{\mathcal{I}}, \Pi_N)$ *is a marginal risk equilibrium for* $(Z_{\mathcal{I}}, Z_N)$ *when* $Z_i = R_{i\omega}(Y_{i\omega}, Y_{-i\omega})$ *for $i \in \mathcal{I}$ and* $Z_N = (V^C_{N,\omega}(Y_{\mathcal{I}\omega}))$.

[15] $U'_\omega(0;1) \ge U'_\omega(q;1)$ for all $q \ge 0$ by concavity of U_ω.

Part 1 of the next result is a direct application of Theorem 7; no proof is needed. Part 2 uses the Cournot Cost Growth Condition from section 4.4, and, like Corollary 3 from that section, is a direct application of Nash's existence theorem [36], hence we omit the proof details.

Theorem 9. *If the Cournot Convexity Assumptions hold,* $\mathcal{W}$ *is a proper subspace of* $\mathcal{Z}$*, and* $\mathcal{D}_0$ *has nonempty interior relative to* $\mathcal{P}$*, then*

1. $(x_{\mathcal{I}}, Y_{\mathcal{I}})$*, with some* $(W_{\mathcal{I}}, W_N)$ *and* $P^{\mathbf{r}}$*, forms a risky Cournot capacity equilibrium with incomplete markets if and only if* $(x_{\mathcal{I}}, Y_{\mathcal{I}})$*, with some* $(\Pi_{\mathcal{I}}, \Pi_N)$*, solves the marginal risky Cournot capacity equilibrium problem; and*

2. *there exists an risky Cournot capacity equilibrium with incomplete markets if, in addition, either the Cournot Cost Growth Assumption holds or the strategy sets* X_i *are bounded.*

See section 4.5.1 for a numerical example that illustrates two incomplete cases of competitive capacity equilibrium models: no risk trading ($\mathcal{W} = \{0\}$) and trading of a single kind of contract ($\mathcal{W}$ has dimension 1).

7 Conclusions

This paper extends capacity expansions problem initially formulated as cost optimization problem during the regulatory period to stochastic equilibrium models aimed at representing markets with an emphasis on imperfections commonly occurring in the energy markets. The objective is to provide a unified computational context sufficiently general to treat different market imperfections that we can be traced to observations (for instance of the market design) or recognized economic market failures (excessive concentration or incomplete risk markets). We separate long and short-term market imperfections so as to progressively construct a catalogue of each (this paper is a first step) that we can combine to assess a wide range of possibilities. Our unifying theory is based on a single formulation of these problems as Nash equilibrium, where the descriptions of the short and long-term markets can be adapted provided they remain in complementarity form. Some of these models reduce to optimization problems but not all. When not, the models also raise non-monotonicity that will need to be explored in the future. In contrast with other work we never get into MPEC and EPEC models because of the numerical issues that they raise and the difficulty of finding an easy economic interpretation of these results. The analysis (existence theorems) is fully explored for a static model and extended to the dynamic set up on the basis of an appropriate change of notation and the restriction of multi-period risk function to guarantee their time consistency.

Numerical results on examples, including the multi-period case, reveal that market imperfections are of definite practical relevance. Investments in the power sector (which are currently stalled except for subsidized capacities) can drastically differ depending on short and long-term market imperfections.

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8 Appendix

8.1 Proof of Theorem 3

As mentioned above, the proof to follow combines Theorems 1 and 2. The bridge between these results is a saddle point characterization of risk averse stochastic programming that we give now.

Lemma 4. *Let X be a nonempty polyhedral convex set in $\mathbb{R}^n$; $\Xi : X \to \mathcal{Z}$ be such that each component function Ξ_ω is be convex and continuous on X; and $r : \mathcal{Z} \to \mathbb{R}$ be a CRM whose risk set $\mathcal{D} \subset \mathcal{P}$ is either polyhedral convex or has nonempty interior relative to $\mathcal{P}$. Then x^* solves $\min_{x\in X} r(\Xi(x))$ if and only if there exists $\Pi^* \in \mathcal{P}$ such that x^* solves $\min_{x\in X} \mathbb{E}_{\Pi^*}[\Xi(x)]$ and Π^* solves $\max_{\Pi\in\mathcal{D}} \mathbb{E}_\Pi[\Xi(x^*)]$.*

Proof: We repeat the discussion at the start of [43, Section 3.2]. Extend each Ξ_ω to take the value ∞ for $x \notin X$. Likewise define the composite functions $r \circ \Xi$ to be $r(\Xi(x))$ when $x \in X$ and ∞ otherwise. Then the subdifferential of $r \circ \Xi$ at $x \in X$ is the set of points $\mathbb{E}_\Pi[\xi]$ where $\Pi \in \partial r(\Xi(x))$ and ξ is an uncertain outcome defined by $\xi_\omega \in \partial\Xi_\omega(x)$ for each ω.

Two notes are that, first, because r is the support function of a nonempty closed convex set $\mathcal{D}$, then for any $Z \in \mathcal{Z}$ its subdifferential $\partial r(Z)$ is the set $\arg\max_{\Pi\in\mathcal{D}} \mathbb{E}_\Pi[Z]$; and, second, the chain rule above applies when $r = \mathbb{E}_\Pi$ for any probability measure Π. Thus for $x \in X$, $\partial(r \circ \Xi)(x)$ is the union of subdifferentials $\partial(\mathbb{E}_\Pi \circ \Xi)(x)$ over $\Pi \in \arg\max_{\Pi\in\mathcal{D}} \mathbb{E}_\Pi[\Xi(x)]$.

We have that x^* solves $\min_{x\in X} r\big(\Xi(x)\big)$ if and only if x^* is a global minimizer of $r \circ \Xi$ if and only if $0 \in \partial(r \circ \Xi)(x^*)$ if and only if there is some $\Pi^* \in \arg\max_{\Pi\in\mathcal{D}} \mathbb{E}_\Pi[\Xi(x^*)]$ such that $0 \in \partial(\mathbb{E}_{\Pi^*} \circ \Xi)(x^*)$. The last inclusion is equivalent to x^* solving $\min_{x\in X} \mathbb{E}_{\Pi^*}[\Xi(x)]$. ■

Proof of Theorem 3: Part 1. The structure of the proof is to show equivalence of each of the following statements using results listed previously:

1. (x_1, x_2) with some $(Y, Q, P, W_1, W_2, P^{\mathrm{r}})$ is a risky competitive capacity equilibrium with complete markets.

2. (x_1, x_2) with some $(P, W_1, W_2, P^{\mathbf{r}})$ is an equilibrium of the following: For $i = 1, 2$, (x_i, W_i) solves

$$\begin{array}{ll} \min\limits_{x_i, W_i} & I_i(x_i) + P^{\mathbf{r}}[W_i] + r_i\big(V_{1\omega}(x_i, P_\omega) - W_{1\omega}\big) \\ \text{subject to} & x_i \in X_i, (9) \text{ for all } \omega, W_i \in \mathcal{W}; \end{array}$$

P_ω clears the spot market in each scenario ω; and $P^{\mathbf{r}}$ clears the risk market (17).

3. (x_1, x_2) with some $(P, P^{\mathbf{r}})$ is an equilibrium of the following: For $i = 1, 2$, x_i solves

$$\min_{x_i} \mathbb{E}_{P^{\mathbf{r}}}\big[V_{1\omega}(x_i, P_\omega)\big] \quad \text{subject to} \quad x_i \in X_i;$$

P_ω clears the spot market in each scenario ω; and $P^{\mathbf{r}}$ solves

$$\max_{\Pi} \mathbb{E}_{\Pi}\big[V_{1\omega}(x_1, P_\omega) + V_{2\omega}(x_1, P_\omega)\big]. \tag{67}$$

4. (x_1, x_2) with some $P^{\mathbf{r}}$ is an equilibrium of the risky Nash game: (x_1, x_2) solves

$$\min_{x_1, x_2} \mathbb{E}_{P^{\mathbf{r}}}\big[V_{0\omega}(x_1, x_2)\big] \quad \text{subject to} \quad x_1 \in X_1,\, x_2 \in X_2; \tag{68}$$

and $P^{\mathbf{r}}$ solves

$$\max_{\Pi} \mathbb{E}_{\Pi}\big[V_{0\omega}(x_1, x_2)\big]. \tag{69}$$

5. (x_1, x_2) solves the risk averse system capacity problem (29).

6. (x_1, x_2) with some (Y, Q) solves the risk averse system capacity problem (28).

$1 \Leftrightarrow 2$ is trivial and immediate, cf., (11) and (12).

$2 \Leftrightarrow 3$ is seen by fixing P and applying Theorem 1 with $\Xi_i : X_i \to \mathcal{Z}$ is defined by

$$\Xi_{i\omega}(x_i) := I_i(x_i) + V_{i\omega}(x_i, P_\omega), \text{ for all } \omega.$$

(The conditions required of Ξ_i are given by Proposition 1.)

$3 \Leftrightarrow 4$: In statement 3, the conditions on x_1, x_2 and P comprise a risk neutral competitive capacity equilibrium model with probability measure $\Pi = P^{\mathbf{r}}$. Given that $P^{\mathbf{r}} \in \mathcal{D}_0 \succ 0$, Corollary 1 says that this equilibrium model is equivalent to (x_1, x_2) solving (68). The equivalence between (67) and (69) uses the fact that $V_{0\omega}(x_1, x_2) = V_{1\omega}(x_1, P_\omega) + V_{2\omega}(x_2, P_\omega)$, for all ω, from Proposition 1.

$4 \Leftrightarrow 5$ is an application of the saddlepoint equivalence in Lemma 4.

$5 \Leftrightarrow 6$: This is elementary. First take a solution $(\hat{x}_1, \hat{x}_2)$ of (29). This induces optimal quantities $\hat{Y}_\omega$, $\hat{Q}_\omega$ for (14) so that $V_{0\omega}(\hat{x}_1, \hat{x}_2) = C_\omega(\hat{Y}_\omega) - U_\omega(\hat{Q}_\omega)$ for each ω. Since

$(\hat{x}_1, \hat{x}_2, \hat{Y}, \hat{Q})$ is feasible for (28), the optimal value of (29) is bounded below by that of (28). Conversely, for any feasible solution (x_1, x_2, Y, Q) of (28), $C_\omega(Y_\omega) - U_\omega(Q_\omega) \geq V_{0\omega}(x_1, x_2)$, hence

$$r_0\big(C_\omega(Y_\omega) - U_\omega(Q_\omega)\big) \geq r_0\big(V_{0\omega}(x_1, x_2)\big)$$

because r_0 is a CRM. This means that the optimal value of (28) is bounded below by that of (29).

Part 2 is derived from the above proof of part 1 via Theorem 1, part 2. ■

The proof of Corollary 2 is the equivalence between statements 1 and 5 in the above proof.

8.2 Proof of Theorem 8, part 2

Proof of Theorem 8, part 2: Take a spot market scenario ω. The Competitive Cost Growth Assumption ensures that the optimization problems in the incomplete risk neutral case (65) are coercive,[16] hence solvable with solution sets bounded in the ∞-norm; indeed for some $c > 0$, its solution sets lie in the ∞-ball of radius c about the origin irrespective of the choice of probability measures Π_1, Π_2.

Note also that for any (x_1, x_2), using concavity of the utility function, the spot market equilibrium $(Y_\omega, Q_\omega, P_\omega)$ is such that $0 \leq P_\omega = U'_\omega(Q_\omega) \leq U'_\omega(0; 1)$ when U_ω is differentiable at Q_ω; and, more generally, P_ω lies in the superdifferential of U_ω at Q_ω with $0 \leq P_\omega \leq U'_\omega(0; 1)$. Define $\mathcal{C}_P$ as the Cartesian product over $\omega \in \Omega$ of the intervals $[0, U'_\omega(0; 1)]$, so that $\mathcal{C}_P$ will contain the stochastic equilibrium price associated with any capacities.

We define three set mappings, one for each type of interaction. First, for the producer and retailer, $i = 1, 2$, let $\Phi_i(P, \Pi_i)$ be the set of optimal solutions (x_1, Y) or (x_2, Q) to the optimization problems in (65). Those problems are continuous and convex with solution sets bounded by c in the ∞ norm, namely, Φ_i maps $\mathcal{C}_P \times \mathcal{P}$ to subsets of S_i where

$$\begin{aligned} S_1 &:= \left\{(x_1, Y) : x_1 \in X_1, \text{ each } Y_\omega \text{ is feasible for (9), and } \|(x_1, Y)\|_\infty \leq c\right\}, \\ S_2 &:= \left\{(x_1, Q) : x_2 \in X_2, \text{ each } Q_\omega \text{ is feasible for (10), and } \|(x_2, Q)\|_\infty \leq c\right\}. \end{aligned}$$

In particular, Φ_i has nonempty convex compact values and also a closed graph. Second, let Φ_P map (x_1, Y, x_2, Q) to the corresponding set of stochastic equilibrium spot market prices $P = (P_\omega)$, i.e., $\Phi_P : S_1 \times S_2 \to \mathcal{C}_P$. This mapping has nonempty set values, which are obviously bounded, from part 3(c) of Competitive Spot Market Assumptions in section 3.1. Indeed $\Phi_P(x_1, x_2)$ is closed and convex because it is the set of KKT multipliers (Proposition 1) of

[16] E.g., the objective function of the producer's problem goes to infinity for feasible (x_1, Y) in each of the three cases: $\|x_1\|_\infty \to \infty$ and $\|Y\|_\infty$ is bounded; $\|x_1\|_\infty \to \infty$ and $\|Y\|_\infty \to \infty$; and $\|Y\|_\infty \to \infty$ which implies the previous case.

a nonlinear program where all functions are convex and continuous on their domains; for the same reason, Φ_P has a closed graph. The final mapping, $\Phi_{\mathrm{r}} : S_1 \times S_2 \times \mathcal{C}_P \to \mathcal{P} \times \mathcal{P}$, sends (x_1, Y, x_2, Q, P) to the set of all marginal risk equilibria (Π_1, Π_2) corresponding to $(Z_1, Z_2) = (V_1(x_1, P), V_2(x_2, P))$. Using Lemma 3 we see that this set mapping has nonempty convex compact values and is also closed since $V_i(x_i, P)$ is continuous on $X_i \times \mathcal{C}_P$.

Kakutani's fixed point theorem now gives us $(x_1^*, x_2^*, P^*, \Pi_1^*, \Pi_2^*)$ such that $x_i^* \in \Phi_i(P^*, \Pi_i^*)$ for $i = 1, 2$, $P^* \in \Phi_P(x_1^*, x_2^*)$, and $(\Pi_1^*, \Pi_2^*) \in \Phi_{\mathrm{r}}(x_1^*, x_2^*, P^*)$. This is a solution of the marginal risky competitive capacity equilibrium problem, and gives existence of a risky competitive capacity equilibrium in incomplete markets by part 1. ■

8.3 Multistage proofs

Proof of Theorem 5: The multistage risky design equilibrium problem and multistage risky Nash game can each be written as single stage problems by translating the multistage space $\mathcal{Z} = \mathbb{R}^K$ to the (single stage) uncertain outcome space $\mathcal{Z}(\mathcal{L}) = \mathbb{R}^L$, hence multistage risky costs $\Xi_i(x_i, x_{-i})$ to (single stage) risky costs $\Xi_i(x_i, x_{-i})^{\mathcal{L}}$; and, via Lemma 2, composite CRMs $r_i = \sigma_{\mathcal{D}_i}$ on $\mathcal{Z}$ to (single stage) CRMs $r_i^{\mathcal{L}} = \sigma_{\mathcal{D}_i^{\mathcal{L}}}$ on $\mathcal{Z}(\mathcal{L})$. Adapting (4), given a price of risk $\hat{P}^{\mathbf{r}}$ on $\mathcal{Z}(\mathcal{L})$, agent i's design problem becomes

$$\min_{x_i, \hat{W}_i} \hat{P}^{\mathbf{r}}[\hat{W}_i] + r_i^{\mathcal{L}}\big(\Xi_i(x_i, x_{-i})^{\mathcal{L}} - \hat{W}_i\big) \quad \text{subject to} \quad x_i \in X_i,\ \hat{W} \in \mathcal{Z}(\mathcal{L}).$$

As usual, at equilibrium $\hat{P}^{\mathbf{r}}$ must ensure that $\sum_i \hat{W}_i = 0$.

The Design Convexity and Design Technical assumptions of Section 2.2 hold trivially for this reformulation, e.g., directional differentiability of $\Xi_i(\cdot, x_{-i})_\omega^{\mathcal{L}}$ holds for $\omega \in \mathcal{L}$ because each of its summands $\Xi_i(\cdot, x_{-i})_{\omega'}$, $\omega' \in [\omega]$, is directionally differentiable. The CRM assumptions, on $\sigma_{\mathcal{D}_i^{\mathcal{L}}}$, hold by Lemma 2. Thus Theorem 1 applies to the single stage versions of the risky equilibrium problem and risky game.

Translating single stage results back to the multistage case requires no additional work. For instance, Theorem 1 ensures that the equilibrium $\hat{P}^{\mathbf{r}}$ lies in $\mathcal{D}_0^{\mathcal{L}}$, i.e., has the single stage form $\Pi^{\mathcal{L}}$ for some multistage probability measure $\Pi \in \mathcal{D}_0$, thus Π is multistage price of risk for the multistage risky design equilibrium problem. ■

Proof of Theorem 6: Consider the multistage risky competitive capacity equilibrium problem with complete markets. To apply Theorem 3 it is convenient to write the equilibrium problem with ξ_i as a decision or control variable and x_i implicitly defined as a state variable, rather than the other way around. To that end we impose nonanticipativity on ξ_i,

$$\text{for all } \omega' \in \mathcal{L}^c,\ \xi_{i\omega} \text{ is the same for all } \omega \in \mathcal{S}(\omega'). \tag{70}$$

Note that such ξ_i generates a stagewise investment $x_i = \big(x_{i\omega'}\big)_{\omega' \in \mathcal{L}^c}$ via

$$x_{i\omega'} := \xi_{i\omega} - \xi_{i\omega'} \quad \text{for each } \omega' \in \mathcal{L}^c \text{ and any } \omega \in \mathcal{S}(\omega').$$

Let $\hat{x}(\xi_i)$ denote this mapping $\xi_i \mapsto x_i$ which allows us to define

$$\begin{aligned} \hat{I}_i(\xi_i) &:= I_i(\hat{x}(\xi_i)), \\ \hat{X}_i &:= \{\xi_i = (\xi_{i\omega})_{\Omega^\circ} : (70) \text{ holds and } \hat{x}(\xi_i) \in X_i\}. \end{aligned}$$

Conversely ξ_i can be recovered from $x_i = \hat{x}(\xi_i)$ via (57)–(58). That is, (x_i, ξ_i) is feasible with respect to $\xi_i \in \hat{X}_i$ and $x_i = \hat{x}(\xi_i)$ if and only if it is feasible with respect to $x_i \in X_i$ and (57)–(58). This allows us to equivalently formulate (59) as

$$\min_{\xi_i, W_i} \hat{I}_i(\xi_i) - P^{\mathbf{r}}[W_i] + r_i\big(V_i(\xi_i, P) - W_i\big) \text{ subject to } \xi_i \in \hat{X}_i. \tag{71}$$

Following the template of the proof of Theorem 5, using Lemma 2, the multistage formulation (71) can be reduced to an equivalent two stage formulation where the first stage is deterministic investment, and the second stage deals with "single stage uncertainties" lying in $\mathcal{Z}(\mathcal{L})$. Agent i, given a price of risk $\hat{P}^{\mathbf{r}}$ for financial products in $\mathcal{Z}(\mathcal{L})$ and a spot price vector P, solves

$$\min_{\xi_i, \hat{W}_i \in \mathcal{Z}(\mathcal{L})} \hat{I}_i(\xi_i) - \hat{P}^{\mathbf{r}}[\hat{W}_i] + \sigma_{\mathcal{D}_i^{\mathcal{L}}}\big(V_i(\xi_i, P)^{\mathcal{L}} - \hat{W}_i\big) \text{ subject to } \xi_i \in \hat{X}_i, \tag{72}$$

where $V_i(\xi_i, P)^{\mathcal{L}}_\omega := \sum_{\omega' \in [\omega]} V_{i\omega'}(\xi_{i\omega'}, P_{\omega'})$ for each $\omega \in \mathcal{L}$, cf., (54).[17] To explain further, formally, in this constructed "single stage" setting, the spot price in a leaf scenario ω is a vector $P_{[\omega]} := (P_{\omega'})_{\omega' \in [\omega]}$ and the corresponding conditions for an equilibrium price are separable, i.e., equivalent to the usual competitive equilibrium price conditions in spot market ω' (laid out in Section 3.1.2) for each $\omega' \in \Omega$. Thus there is no loss of generality in using $P = (P_\omega)_{\omega \in \Omega}$ to represent equilibrium prices even though, formally, $P_{[\omega]}$ is unrelated to the price at another leaf node.

Thus the translation from multiple stages to two stages yields a two stage risky competitive capacity equilibrium with complete markets. Part 1 of Theorem 3 says this is equivalent to the (single stage) risk averse optimization problem

$$\begin{aligned} \min_{\xi_1, \xi_2} \quad & \hat{I}_1(\xi_1) + \hat{I}_2(\xi_2) + \sigma_{\mathcal{D}_0^{\mathcal{L}}}\big(V_0(\xi_1, \xi_2)^{\mathcal{L}}\big) \\ \text{subject to} \quad & \xi_1 \in \hat{X}_1,\ \xi_2 \in \hat{X}_2. \end{aligned}$$

This problem is equivalent to (60) because $\sigma_{\mathcal{D}_0}(Z) = \sigma_{\mathcal{D}_0^{\mathcal{L}}}(Z^{\mathcal{L}})$ from Lemma 2.

■

[17] E.g., in the simplest capacity setting, in scenario $\omega \in \mathcal{L}$, the generator is given the price $(P_{\omega'})_{[\omega]}$ and capacity $(\xi_{1\omega'})_{[\omega]}$, and sets production $0 \le (y_{\omega'})_{[\omega]} \le (\xi_{1\omega'})_{[\omega]}$ to yield its optimal cost $V_1(\xi_1, P)^{\mathcal{L}}_\omega$.